THE

MATHEMATICAL GAZETTE

EDITED BY

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"THE WONDERFUL CANON OF LOGARITHMS."

"MORE than fifty years ago, when a boy at school, I first saw a 7-figure table of logarithms, and was shown how to use it. My feeling of amazement at what it could do is a vivid memory to me. . . . Although in all the years since then I have been interested in, and concerned with, mathematical tables and logarithmic computations, I have only admired more and more the table itself as a consummate piece of human effort, applying with singular effect mathematical principles to the general service of mankind." These are J. W. L. Glaisher's words, in the magnificent *Napier Tercentenary Memorial Volume* (1915) edited for the Royal Society of Edinburgh by C. G. Knott; many of us must have experienced the same feeling of awe on first encountering the almost supernatural power of calculation granted to the user of logarithms.

John Napier of Merchiston (1550-1617) might well entitle his first public announcement of this potent instrument *Mirifici Logarithmorum Canonis Descriptio*. Published in 1614, after labours which appear to have occupied him for some twenty years, it was translated into English in the following year by Edward Wright, and followed in 1619 by the posthumously-published *Constructio*, wherein the method of calculation was explained.

The *Napier Memorial Volume*, mentioned above, is an indispensable storehouse of information on all matters concerning Napier and the invention of logarithms. Some further biographical details are discussed in articles in the *Gazette* by W. R. Thomas (XIX, July, 1935) and A. Inglis (XX, May, 1936). For the plate of the title-page of the 1614 *Descriptio*, we are indebted to the Photographic Department of the British Museum.

THE MATHEMATICAL ASSOCIATION.

Annual Meeting, 1950.

The Annual General Meeting of the Mathematical Association was held at the Polytechnic, Regent Street, London, W. 1, on 3rd and 4th January, 1950.

On 3rd January the business meeting was held at 10.30 a.m., with the President, Mr. A. Robson, in the chair. The Report of the Council for 1949 was adopted, and a report on the financial situation by the Treasurer was received.

The election of Professor H. R. Hassé as President for 1950 was announced. The existing Vice-Presidents, the Treasurer, the Secretaries, the Librarian, the Editor of the *Mathematical Gazette*, and the Auditor were re-elected. The following members were elected to serve on the Council: Dr. F. G. Maunsell, Mr. M. A. Porter, Professor T. Arnold Brown, Mr. C. T. Daltry, Dr. J. Topping, Miss F. Pendry, Mr. W. J. Langford, Professor R. L. Goodstein, Mr. D. B. Bousfield.

Sir Harold Spencer Jones gave his Presidential Address (postponed from 1949) on " The measurement of time ".

The remaining items of the programme were:

3rd January.

2 p.m. Discussion on 'Mechanics in Schools ", opened by Mr. K. S. Snell, Dr. E. A. Baggott and Miss L. E. Hardcastle.

5 p.m. " The Method of Mathematical Physics " by Professor H. S. W. Massey, F.R.S.

4th January.

10 p.m. " Abstract Analysis " by Dr. F. Smithies.

11.15 p.m. " Automatic Calculating Machines " by Professor D. R. Hartree, F.R.S.

2 p.m. Discussion on " Mathematics in the Comprehensive School ", opened by Mr. F. J. Swan and Miss Y. Guiseppi.

5 p.m. " A Theory of Measurement " by Professor H. Dingle.

A Publisher's Exhibition was open throughout the meeting.

REPORT OF THE COUNCIL FOR THE YEAR 1949

Membership.

During the period from 1st November, 1948, to 31st October, 1949, 183 full and 36 Junior members were admitted. The membership at 31st October, 1949, was 2,679, of whom 7 are Honorary, 216 Life Members, 2,190 Ordinary Members and 266 Junior Members. This represents an increase of 109, which is an encouraging number. The Council reports with regret the death of 11 members during this period, including Mr. W. N. Stocker who, having joined in 1887, was the senior member of the Association.

Other members who have died during this period are Mr. T. G. Strain (1913), Prof. E. M. Wellish (1916), Mr. A. Buxton (1919), Mr. J. W. Brooks (1921), Prof. H. R. Hamley (1931), Rev. S. H. Semple (1924), Mr. W. Hunter (1936), Miss E. M. Smith (1943), Mr. C. E. Kemp (1946) and Mr. C. H. Clayton (1947).

Finance.

The balance in hand on 1st November, 1948, was £251 8s. 7d., and on 1st November, 1949, was £279 2s. 3d. Receipts include £502 0s. 3d. for the sale of the remaining Defence Bonds, so that the actual loss for the year has been £474 6s. 7d.

The estimated loss was £850, and the actual figures for most items show a very close agreement with those upon which the Committee appointed by the Council based its recommendations for increased subscriptions. The bill for *Gazette* No. 305, which was included in the estimates, has not come through in time for payment in the accounting period; had it done so the actual loss would have been very near to the estimate, as the cost of another *Gazette* would be in the neighbourhood of £350.

The most important result is that it has not been necessary to sell any of the £1,100 War Loan which is now the only remaining financial reserve of the Association.

It is felt that the Association can expect a reasonable surplus in the next accounting period and that the Reports which are now in course of preparation will not be held up by lack of funds. The future is reasonably secure, provided only that the present membership does not fall because of the increased subscriptions; the Branches are again asked to continue their excellent efforts in securing new members.

The Equalisation Fund, which was established last year through the generosity of two of the larger Branches, has been of assistance to two recently formed Branches, to whom payments totalling £7 6s. have been made.

The Branches.

Two new branches have been formed during the year—the North Lancs and Westmorland Branch, which was fortunate in receiving the assistance of Mr. Robson in the inaugural stages, and the North Staffordshire and District Branch, for whose formation Mr. Vesselo has been largely responsible. We wish these branches every success, and hope that they will receive full support from Association members in their localities.

The success of the Birmingham meeting reflects great credit on the initiative and organizing ability of the Midland Branch, and a number of provincial branches are making plans for extending their hospitality to the Association in future years. The Bristol Branch is already taking steps to demonstrate that provincial hospitality is not confined to one area.

Keen interest continues to be shown in branch activities, and membership continues to rise in a most satisfactory manner. London has had some successful meetings of Secondary Modern School members, and hopes to develop further contacts in this direction. Southampton Branch has held meetings at Winchester, and Plymouth and District holds meetings in different parts of its scattered area. Programmes of other branches reveal a fine catholicity of interest catering for a wide range of mathematical ability.

Overseas branches also continue to flourish. Reports of activities of Australian members have been printed in the *Mathematical Gazette*, and frequent expressions of goodwill and interest in the Association have been received from Victoria, Sydney and Queensland Branches.

At Auckland, a Refresher Course for Mathematics Teachers is being held in January, 1950, when publications of the Association and lectures by our members will have a prominent place.

The Mathematical Gazette.

Volume XXXIII will be completed in four parts instead of five, though the total size of the volume is not reduced. This effects an economy in cost of distribution which will be continued for the present time of financial stringency.

The increasing proportion of review pages is to some extent a measure of the gradual return to the pre-war level of production of mathematical textbooks and treatises.

The Teaching Committee.

Normal sub-committee activity has continued throughout the year, and the whole Teaching Committee met at Birmingham in April following the Annual General Meeting of the Association. At that meeting reports were received from all sub-committees, and full consideration was given to the new report on the Teaching of Calculus.

The pamphlet on *Mathematics in Secondary Modern Schools* has been published and reached all members of the Association with the October *Gazette*. The two reports prepared by the Technical Schools sub-committee have also been issued ; the full Trigonometry report has passed the first-proof stages, and it is hoped that, if binding does not cause undue delay, this long-awaited report will be issued to members in 1950. The Calculus sub-committee under the Chairmanship of Mr. Snell, with Mr. Robson as Editor and Mr. Prag as Secretary, produced a report which was accepted unanimously by the Teaching Committee, and Council has consented to publication.

The other sub-committees—on Geometry in the Sixth Form, on Visual Aids, on Mathematics in Technical Colleges and on Mathematics in Primary and Secondary Modern Schools—have continued steadily with their work throughout the year.

This year brings to an end the four-year term of office of the present Teaching Committee and its successor will meet for the first time on Thursday, 5th January, 1950. According to present regulations the new Committee will consist of 48 members, of whom 15 have been appointed by the retiring Committee from its own membership. Council has appointed the remainder, and, while the condition that at least 15 members must be new to the Committee has been liberally fulfilled, care has been taken to ensure that the work of sub-committees, at present engaged on reports, has been safeguarded.

Problem Bureau.

Many of the applicants during this year are members who have only recently joined the Association.

A great variety of questions have been sent in, Mechanics from Higher School Certificate papers being perhaps the most usual.

Officers and Council.

The Council wishes to express its sincere thanks to Mr. Robson for the able manner in which he has fulfilled his duties during his tenure of the office of President.

The Council wishes also to record appreciation of the work of the officers. From Editor, Librarian, Treasurer and Secretaries alike the services rendered have been of the high standard that the Association has learned to expect. So also have those of the members of the Problem Bureau and of the various committees. It is particularly appropriate to mention the names of Mr. W. J. Langford and Dr. J. Topping, who have been, respectively, Chairman of the Teaching Committee and Programme Secretary, and who now find it necessary to resign from these positions, after performing most valuable work.

The thanks of the Council are also due to the retiring members, Mrs. C. M. Williams and Dr. E. A. Maxwell.

ANNUAL MEETING, 1951.

THE Annual General Meeting for 1951 will be held at **Bristol** on 28th, 29th, 30th and 31st March, 1951. The meetings will be held in the University, and accommodation will be available for members and their wives or husbands at Wills Hall, one of the University's Hostels. Application forms will be circulated to members towards the end of 1950.

REPORT OF THE MEETING OF THE TEACHING COMMITTEE.

5th January 1950.

THE Teaching Committee is appointed every four years at the Annual General Meeting; the present Committee was so appointed on 3rd January, 1950. A list of names of members is given below. The regulations require that at least fifteen members of the new Committee shall not have served on the retiring Committee. Consequently the first meeting after appointment, and the first subsequent meetings of sub-committees, provide occasions for reviewing policy and progress and for making new starts.

Mr. W. J. Langford, the retiring Chairman, having expressed a wish not to serve in that capacity during the new period, the Committee recorded their warm appreciation of his services during the past four years, and elected Mr. J. T. Combridge as his successor.

Mr. J. C. Manisty was re-elected Secretary of the Committee, and Mr. M. W. Brown was elected Assistant Secretary.

The Standing Sub-committee is authorised to fill vacancies on the Teaching Committee, and at the time of writing this report two vacancies remain.

Sub-committees were constituted as shown by the reference letters against the names of members in the list below. The task of compiling the Report on the Teaching of Sixth Form Geometry, hitherto carried out by a section of the Sixth Form Sub-committee, was allotted to a separate sub-committee. This action leaves the new Sixth Form Sub-committee free to consider a new Report on the Teaching of Sixth Form Algebra, and to investigate the relations between Sixth Form Mathematics in Schools and First Year Mathematics in the Universities.

An interesting discussion showed that there is a need for helping all who have to teach Mathematics to understand better the relations of the various portions of the subject (however elementary) to the curriculum and to the subject as a whole, and to appreciate more fully the reasons why certain methods of presenting particular topics are preferable to others. As a result, the Standing Sub-committee was asked to consider whether a sub-committee should be appointed to investigate this matter, and, if thought desirable, to appoint such a sub-committee and to consult the Branches. All sub-committees were asked to consider the same matter, and to send any recommendations to the Standing Sub-committee. (The latter has since met, and has appointed a Professional Training Sub-committee, indicated in the list below, with Mr. M. W. Brown as convener. It is hoped that this sub-committee will make an early start on the preparations for a task which is likely to be complicated and arduous.)

The Secretaries of the 1946–50 sub-committees presented progress reports. The Trigonometry Report was in process of being printed; the Calculus Report required no further meetings of its sub-committee. Reports on *Mathematics in the Secondary Technical School* and on *The Training of Teachers of Mathematics in Technical Schools and Colleges* had been issued with the October *Gazette*, and also an interim report on *The Teaching of Mathematics in Secondary Modern Schools*. For Primary Schools a considerable amount of material had been collected and a drafting team appointed. Both the Sixth Form Geometry and the Visual Aids sub-committees had several chapters of their Reports complete; the former hoped to finish in about a year's time, while the latter was making steady progress with a Report which is likely to be of more than average size.

It cannot be too strongly emphasised that for the success of these Reports (which are a very important part of the work of the Association) we need not only the efforts of those who compile the drafts but also suggestions and

contributions from all members of the Association who have any views on the subject-matter. Many who have something which they feel is not important enough for the *Gazette* could help by sending a note with some suggestions to the Secretary of the appropriate sub-committee, or, if he is not known, through the Secretary of the Teaching Committee.

J. T. C.

Teaching Committee 1950.

Ex-officio			
	*	The President	Professor H. R. Hassé (Bristol University).
a	f	The Hon. Treasurer	Mr. J. B. Morgan (Harrow School).
d		The Hon. Secretaries	Mr. F. W. Kellaway (North Herts Tech. C.)
b			Miss M. E. Bowman (Maria Grey T.C.).
	f *	The Editor	Prof. T. A. A. Broadbent (R.N.C., Greenwich).
Universities			
	f	Dr. I. W. Busbridge	St. Hugh's College, Oxford.
a		Miss M. L. Cartwright	Girton College, Cambridge.
	f	Miss W. L. C. Sargent	Bedford College, London.
d		Mr. B. H. Chirgwin	Queen Mary College, London.
	*	Mr. J. T. Combridge	King's College, London (*Chairman*).
d e		Dr. F. G. Maunsell	University College, Southampton.
a		Dr. E. A. Maxwell	Queens' College, Cambridge.
a		Prof. E. H. Neville	Reading University.
Training Colleges			
e		Miss O. Compton	Trent Park T.C., Barnet.
bc	g*	Mrs. E. M. Williams	City of Leicester T.C.
c		Mr. R. H. Cripwell	Didsbury T.C., Manchester.
	g	Mr. C. T. Daltry	Institute of Education, London.
b e		Mr. B. J. F. Dorrington	Camden T.C., London.
Technical Colleges and Schools			
d		Miss I. P. Rosser	South East Essex Technical College.
d		Mr. A. J. L. Avery	Derby Technical College.
d		Mr. H. V. Lowry	Woolwich Polytechnic.
d		Mr. C. G. Paradine	Battersea Polytechnic.
Secondary Schools			
e		Miss H. Bromby	Southampton G.S. for Girls.
c	f	Miss W. Garner	Whalley Range High School, Manchester.
b		Miss V. Y. Guiseppi	West Norwood Secondary School.
	f	Miss E. M. Read	formerly King's Norton High School.
b		Miss K. M. Sowden	Speedwell Secondary Modern School, Bristol.
a	g	Mr. W. Armistead	Christ's Hospital, Horsham.
b	g*	Mr. M. W. Brown	Peckham Secondary School (*Asst. Secretary*).
a		Mr. C. V. Durell	formerly Winchester College.
b	g	Mr. K. R. Imeson	Sir J. Williamson's Mathematical School, Rochester.
	f *	Mr. W. J. Langford	Battersea Grammar School.
	*	Mr. J. C. Manisty	Winchester College (*Secretary*).
e		Mr. D. G. R. Martin	William Hulme's G.S., Manchester.
	f	Mr. G. L. Parsons	Merchant Taylors' School.
	f	Mr. A. Robson	formerly Marlborough College.
	f	Mr. K. S. Snell	Harrow School.
a		Mr. C. O. Tuckey	formerly Charterhouse.

Preparatory and Primary Schools

b c	Miss H. M. Cook	Edge Hill T.C., Ormskirk.
c	Miss R. E. Morris	Dinorben School, Wallington.
c	Mr. G. Ashton	Abberley Hall School, Worcester.
b c d	Mr. A. W. Riley	Inspectorate, Wolverhampton.
	Two vacancies.	

Special Interests

f	Mr. A. Prag	Westminster School.
b	Mr. J. C. Skinner	Inspectorate, Bradford.
b g	Mr. C. W. Tregenza	H.M.I., Ministry of Education.
b d e	Mr. I. R. Vesselo	Alsager Training College.

Sub-committees

a Sixth Form Geometry.
b Modern Schools.
c Primary Schools.
d Technical Colleges.
e Visual Aids.
f Sixth Form.
g Professional Training.
* Standing Sub-committee.

MATHEMATICAL COLLOQUIUM, OXFORD, 1950.

The second British Mathematical Colloquium was held in Oxford on 12th, 13th and 14th April, 1950. About 90 members were accommodated in Jesus College, and about 40 others were present but did not stay in college.

The programme divided itself into two main parts, the reading of lectures and papers, and the more informal discussions at the numerous "splinter groups". Lectures and papers filled the mornings and the intervals between tea and dinner:

12th April.

9.30 a.m. "Algebraic Methods in Analysis" by Dr. F. Smithies. "Abstract Proofs of Tauberian Theorems" by Dr. J. L. B. Cooper. "An Eigen-function Problem" by Professor E. C. Titchmarsh.

5 p.m. "Some Boundary Value Problems" by Dr. M. L. Cartwright.

13th April.

9.30 a.m. "Valuation Theory" by Dr. K. Mahler (illness prevented Dr. Mahler from attending, and his paper was read by Dr. B. H. Neumann). "Applications of Valuation Theory to the Theory of Ideals" by Dr. D. G. Northcott. "Valuation Theory and Birational Geometry" by Dr. D. B. Scott.

5 p.m. "Cohomology in Abstract Algebra" by Mr. D. Rees.

14th April.

9.30 a.m. "Topological Groups" by Professor M. H. A. Newman. "The Hilbert Problem for Three-Dimensional Groups" by Mr. P. J. Hilton. "Fibre Mappings" by Professor J. H. C. Whitehead.

5 p.m. "Fibre Bundles" by Mr. R. Taylor.

In addition, on the afternoon of 14th April, the Printer to the University (Mr. C. Batey) gave a talk on mathematical printing, which was followed by a visit to the Clarendon Press.

The "splinter groups" on algebra, algebraic geometry, theory of numbers, etc., met at afternoon and after-dinner sessions, and displayed tremendous enthusiasm and energy of discussion. The present writer has, however, no first-hand evidence to enable him to decide whether a meeting of one group, called for 6 a.m. on 15th April, did take place.

The chief impression left by the Colloquium is that British mathematics is very much alive, tackling its problems with vigour and skill, and not suffering from the old British fault of insularity.

AN OPERATIONAL METHOD FOR DETERMINING THE SERIES SOLUTION OF A LINEAR DIFFERENTIAL EQUATION OF RANK TWO.

By K. Sarginson.

Introduction.

The equation considered is of the type

$$[f(\delta) - xg(\delta)]y = 0, \qquad \text{(1)}$$

where

$$\delta \equiv x\frac{d}{dx},$$

and $f(\delta)$ and $g(\delta)$ are polynomials in δ.

There are three cases to consider. These are :

(I) The roots of $f(z) = 0$ are distinct and no two differ by an integer.

(II) Some of the roots of $f(z) = 0$ are co-incident.

(III) There exists an integral difference between some of the roots of $f(z) = 0$.

Case I.

Let $f(\delta) \equiv \prod_{r=1}^{n} (\delta - c_r)$, where all the c_r are distinct and no two differ by an integer.

The differential equation can be written

$$f(\delta)\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]y = 0,$$

i.e.

$$\prod_{r=1}^{n} (\delta - c_r)\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]y = 0. \qquad \text{(2)}$$

$$\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]y = \sum_{r=1}^{n} K_r x^{c_r},$$

where $K_1, K_2 \ldots K_n$ are arbitrary constants. Thus

$$y = \sum_{r=1}^{n}\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]^{-1} K_r x^{c_r}, \qquad \text{(3)}$$

where $\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]^{-1}$ is defined by means of

$$\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]^{-1} \equiv 1. \qquad \text{(4)}$$

This condition is fulfilled by

$$\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]^{-1} = \sum_{s=0}^{\infty} x^s \frac{[g(\delta)]_s}{[f(\delta)]_s}, \qquad \text{(5)}$$

with the notation $[g(\delta)]_s = \prod_{t=1}^{s} g(\delta + t - 1)$, $s > 1$, $[g(\delta)]_0 = 1$. Thus

$$y = \sum_{r=1}^{n} K_r x^{c_r} \sum_{s=0}^{\infty} x^s \frac{[g(c_r)]_s}{[f(c_r+1)]_s}, \qquad \text{(6)}$$

the solution being valid for values of x which make the series converge.

Lemma. The following lemma will be used in solving the equation in cases II and III.

When $f(\delta)$ can be expressed in a series involving positive integral powers of δ only, then the value of $f(\delta)(\log_e x)^r$ is $\sum_{s=0}^{r} {}^r c_s f^{(s)}(0)(\log_e x)^{r-s}$.

$$f(\delta)(\log_e x)^r = \sum_{s=0}^{\infty} \frac{f^{(s)}(0)\delta^s}{s!}(\log_e x)^r = \sum_{s=0}^{r} \frac{f^{(s}(0)}{s!}\,\frac{r!}{(r-s)!}(\log_e x)^{r-s},$$

since

$$\delta = x\frac{d}{dx} = \frac{d}{d(\log_e x)}.$$

A similar result can be obtained for a particular integral of $f(\delta)\,.\,(\log_e x)^r$ when $f(\delta)$ involves negative powers of δ, but it will not be required.

Case II. Suppose $f(z)=0$ has roots c_r repeated m_r times, $r=1, 2, \ldots n$, but no two c_r differ by an integer. The differential equation can then be written

$$\prod_{r=1}^{n} (\delta - c_r)^{m_r}\left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]y = 0. \quad \text{.......................(7)}$$

The general solution of $(\delta - c)^m u = 0$ is $u = \sum_{s=0}^{m-1} K_s(\log_e x)^s x^c$, where $K_0, K_1 \ldots K_n$ are arbitrary constants.

So a solution of (7) is

$$\begin{aligned} y &= \left[1 - x\frac{g(\delta)}{f(\delta+1)}\right]^{-1} \sum_{r=1}^{n}\sum_{s=0}^{m_r} K_{rs}x^{c_r}(\log_e x)^s \\ &= \sum_{r=1}^{n}\sum_{s=0}^{m_r}\sum_{t=0}^{\infty} K_{rs}x^t \frac{[g(\delta)]_t}{[f(\delta+1)]_t}x^{c_r}(\log_e x)^s \\ &= \sum_{r=1}^{n}\sum_{s=0}^{m_r}\sum_{t=0}^{\infty} K_{rs}x^{c_r+t}\frac{[g(\delta+c_r)]_t}{[(\delta+c_{r+1})]_t}(\log_e x)^s \\ &= \sum_{r=1}^{n}\sum_{s=0}^{m_r} K_{rs}x^{c_r}\sum_{t=0}^{\infty} x^t \sum_{p=0}^{s} {}^sC_p F_t^{(p)}(0)(\log_e x)^{s-p}, \quad \text{...............(8)} \end{aligned}$$

where $$F_t(z) = \frac{[g(z+c_r)]_t}{[f(z+c_r+1)]_t}.$$

This is valid provided the infinite series converge and provided no term in $[f(\delta+c_r)]_t$ has a factor δ. The latter is the case since no two c_r differ by an integer.

Case III. Suppose now that $f(z)=0$ has two or more roots which differ by an integer. In this case the equation can be reduced to one of the type where $f(z)$ has equal roots. I shall consider the special case when $f(z)$ is of the second degree. The method can be extended readily to the general case.

Consider then the equation

$$[\delta(\delta - n) - xg(\delta)]y = 0, \quad \text{...............................(9)}$$

when n is a positive integer.

Let $y = \delta(\delta-1)\ldots(\delta-n+1)z$, where for given y, z is a particular integral of $\dfrac{1}{\delta(\delta-1)\ldots(\delta-n+1)}y$. Thus we require z to involve two arbitrary constants only. The equation for z is

$$(\delta-1)\ldots(\delta-n)[\delta^2 - xg(\delta)]z = 0;$$

and sufficiently

$$[\delta^2 - xg(\delta)]z = 0,$$

which has a solution

$$z=\sum_{r=0}^{\infty} x^r \frac{[g(\delta)]_r}{[(\delta+1)^2]_r}\,[K_0+K_1\log_e x].$$

Thus
$$y=\sum_{r=0}^{\infty} x^r \frac{(\delta+r-n+1)_n[g(\delta)]_r}{[(\delta+1)^2]_r}\,[K_0+K_1\log_e x]$$

$$=K_0\sum_{r=0}^{\infty} F_r(0)x^r+K_1\sum_{r=0}^{\infty} x^r\{F_r(0)\log_e x+F_r'(0)\},$$

where
$$F_r(z)=\frac{(z+r-n+1)_n[g(z)]_r}{[(z+1)^2]_r}.$$

Now $F_r(0)=0$ for $r=0, 1, 2, \dots, (n-1)$. So we can write the solution

$$y=K_0\sum_{r=0}^{\infty}\frac{[g(n)]_r}{(r+n)!\,r!}\,x^{r+n}+K_1\sum_{r=0}^{\infty} x^r\phi_r(x),$$

where $\phi_r(x)$ is the value of $\dfrac{\partial}{\partial c}[x^cF_r(c)]$ when c is put equal to zero after differentiation. A factor $[g(0)]_n$ has been incorporated in K_0, and therefore we have to make the provision that $g(z)$ does not contain a factor $(z-m)$ where m is an integer and $0\leqslant m\leqslant(n-1)$, for in this case $[g(0)]_n$ would be identically zero. When $g(z)$ contains such a factor, the infinite series

$$\sum_{r=0}^{\infty}\frac{[g(n)]_r}{(r+n)!\,r!}\,x^{r+n}=y_1$$

is still a solution of the equation. $F_r(0)$ vanishes for all values of r and $F_r'(0)$ vanishes when $(m+1)\leqslant r\leqslant(n-1)$. So the term $\sum_{r=0}^{\infty} x^r[F_r(0)\log_e x+F_r'(0)]$ reduces to $\sum_{r=0}^{m} x^rF_r'(0)+\sum_{r=n}^{\infty} x^rF_r'(0)$. When $r\geqslant 0$, $F_{n+r}'(0)$ is a constant multiple of $\dfrac{[g(n)]_r}{(r+n)!\,r!}$, so the infinite series is just a multiple of y_1. The multiplier is zero if $g(z)$ contains more than one factor $(z-m)$ for $0\leqslant m\leqslant n-1$, but in any case y_1 is already obtained as a solution. The value of y is thus

$$y=K_0\sum_{r=0}^{\infty}\frac{[g(n)]_r}{(r+n)!\,r!}\,x^r+\sum_{r=0}^{m} x^rF_r'(0),$$

where m is the least integral zero of $g(z)$ between 0 and $(n-1)$ inclusively.

When $f(z)$ is of degree higher than the second and it contains more than one pair of factors which differ by an integer the method of procedure is similar to the above. Suppose, for example, $f(z)=0$ has roots a_1, a_1+n_1; a_2, a_2+n_2; ..., where n_1, n_2 ... are integers. The substitution

$$y=[(\delta-a_1)(\delta-a_1-1)\dots(\delta-a_1-n_1+1)][(\delta-a_2)(\delta-a_2-1)$$
$$\dots(\delta-a_2-n_2+1)]\dots z$$

reduces the equation to one of the type considered in II.

When $f(z)=0$ has a set of, say, n roots, the difference between any pair being integral then, in general the solution of the differential equation will involve powers of $\log_e x$ up to the $(n-1)$th.

In conclusion, I would like to thank Mr. T. W. Chaundy for his advice and suggestions regarding this note.

K. S.

SOME NOMOGRAMS FOR THE PILOT NAVIGATOR.

By J. C. Cooke.

A pilot flying solo is in a somewhat difficult position as regards air navigation. Too much in my opinion has been written about how to navigate an aeroplane for the man with an array of instruments and a comfortable chair and chart table to work at, and not enough for the other poor unfortunate. The pilot navigator (always excepting people like Mollison or Lindbergh, Amy Johnson or Jean Batten) is limited, and has to find his way about mostly by map-reading, but if he does his job properly he should at least find out the wind from the weather people, and use it to calculate his course to steer and his ground speed. These will never be quite right, but will be near enough for him to start off; he can check how he is going by his map and make the necessary corrections.

The problem is: Given the wind speed and direction, the air speed and desired track, to find the course to steer and the ground speed. Here we take as given the air speed, the wind speed and the angle between the wind and the desired track, to find the drift and ground speed; *i.e.* given a, w and ϕ, to find d and g. (See Fig. 1.) It is, of course, the ambiguous case in

Fig. 1.

the solution of triangles, but there is no ambiguity in practice. ϕ is taken from 0° to 90° (tail wind),* and from 0° to 90° (head wind).

Many methods are known and many instruments have been devised to solve this problem, usually too cumbrous for the pilot navigator, yet I have never seen in print the simplest of all if you have a slide rule. Reverse the slide so that the S scale (of sines) is opposite the top scale. Set the angle ϕ against the given air speed, and read off the *drift* opposite the given wind speed. Add (subtract) this drift to the original ϕ if the wind is a tail (head) wind, and read off the *ground speed* against this new angle.

My idea here was to make two nomograms which, while covering a reasonable range of air speeds and wind speeds, were to fit into the case of a Douglas protractor (about $5\frac{1}{2}''$ square). I did not want to have to use a ruler and pencil, which is fiddling (often infuriating) in the air, but to be able to follow

* A wind from any direction abaft the beam is here called a "tail wind", from any direction forward of the beam a "head wind".

lines with my finger. I did not succeed in this in the case of finding the ground speed, but at least I did not have first to use a ruler, then make a pencil mark and then use the ruler again, as is often expected. The ranges chosen were from 100 to 250 m.p.h. for air speed, and from 0 to 50 m.p.h. for wind speed. When constructed these two nomograms were stuck one on each side of a piece of thin cardboard and slid into the case of the protractor.

The Determination of Drift.

We have $\sin d = (w/a) \sin \phi$, and the first step is to determine w/a.

We measure a distance of $1/a$ along the line Ox (see Fig. 2) * and draw a series of lines OB through O representing different wind speeds. Then the

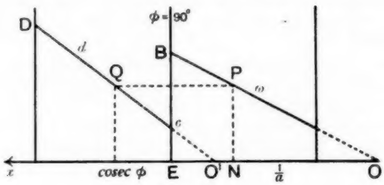


FIG. 2.

ordinate PN represents w/a if the slope of OB is equal to the magnitude of w. Along $O'x$ we measure a distance $\operatorname{cosec} \phi$, starting at $\phi = 90$ (point E) and going down by (say) $10°$ at a time. We cannot of course go all the way down to $\phi = 0$; this will be considered below. Then the coordinates of Q are $(\operatorname{cosec} \phi, w/a)$ referred to O' as origin, and as $(w/a)/\operatorname{cosec} \phi = \sin d$ the slope of the line CD ($d =$ const.) will be $\sin d$.

Hence in order to find d we start at the point N, go vertically upwards to meet the wind line at P, horizontally to cut the vertical through ϕ at the point Q which lies on one of the drift lines d.

The snag is that the graduations on the line Ex are cramped near E, and yet they stretch out indefinitely along the line to the left. This was avoided in the actual nomogram constructed by graduating Ex evenly in terms of ϕ and going from $\phi = 90°$ to $\phi = 10°$. The lines CD then become curves asymptotic to the vertical through $\phi = 0$. These curves were drawn at intervals of $2°$ for d. The drawing was done on squared paper; verticals at intervals of 10 m.p.h. for air speed and $10°$ for ϕ were inked in. Part of the drawing is shown in Fig. 3, which gives the dimensions used and the solution of a problem in dotted lines.

The graduations of a are cramped and possibly difficult to read for a lying between 200 and 250. It might have been better for clarity if a had been graduated evenly; the w lines would then have been hyperbolas and would have been quite simple to construct. I did not in fact do this, but nevertheless found that the nomogram worked well in practice. In this region an error of (say) 10 m.p.h. in a makes little difference to the drift. Very few pilots can steer a course to within an accuracy of one degree. I doubt if the pilot's compass can be *set* even to an accuracy of one degree.

The Determination of Ground Speed.

The relation here is

$$a^2 = g^2 + w^2 - 2gw \cos \phi. \qquad \text{(1)}$$

Taking origin at O (see Fig. 4), using oblique axes Ox, Oy, graduate $\cos \phi$ along Oy and a^2 along AB. Then the equation of the line QR is

* To a suitable scale. This remark applies to all graduations.

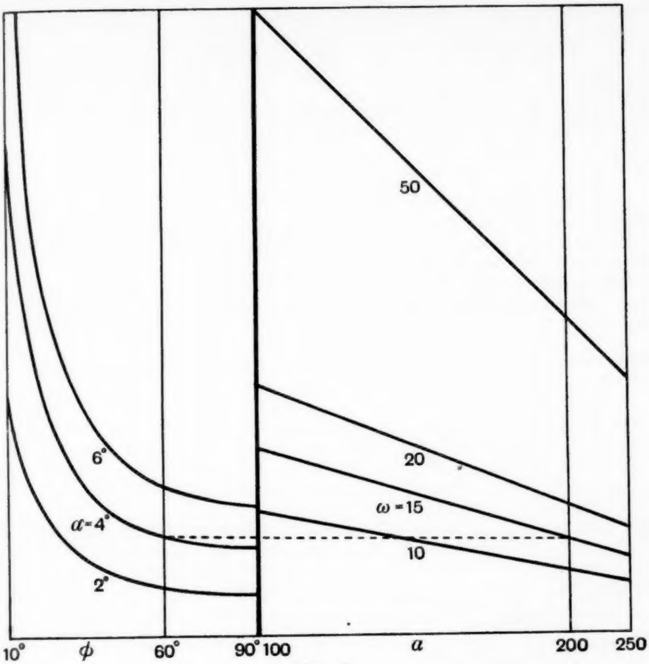


FIG. 3.

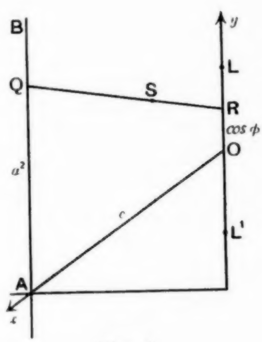


FIG. 4.

$$y = \frac{a^2 - \cos\phi}{c}\,x + \cos\phi.$$

Using (1) we may rewrite this equation as

$$y-\frac{g^2+w^2}{2gw+1}=\frac{g^2+w^2-(2gw+1)\cos\phi}{c}\left(x-\frac{c}{2gw+1}\right)$$

which shows that, provided (1) is satisfied, the line QR (for fixed g and w) always passes through the fixed point S of coordinates

$$\left(\frac{c}{2gw+1},\ \frac{g^2+w^2}{2gw+1}\right).$$

Hence we plot a network of curves $g=$ const., $w=$ const., and to solve any problem given a, w and ϕ we join the given points Q and R; QR cuts the given w curve on one of the g curves, thus giving g.

It will easily be seen that $g=$ const., $w=$ const., are all parts of hyperbolas, in fact by symmetry they all belong to one family; and are all asymptotic to Oy.

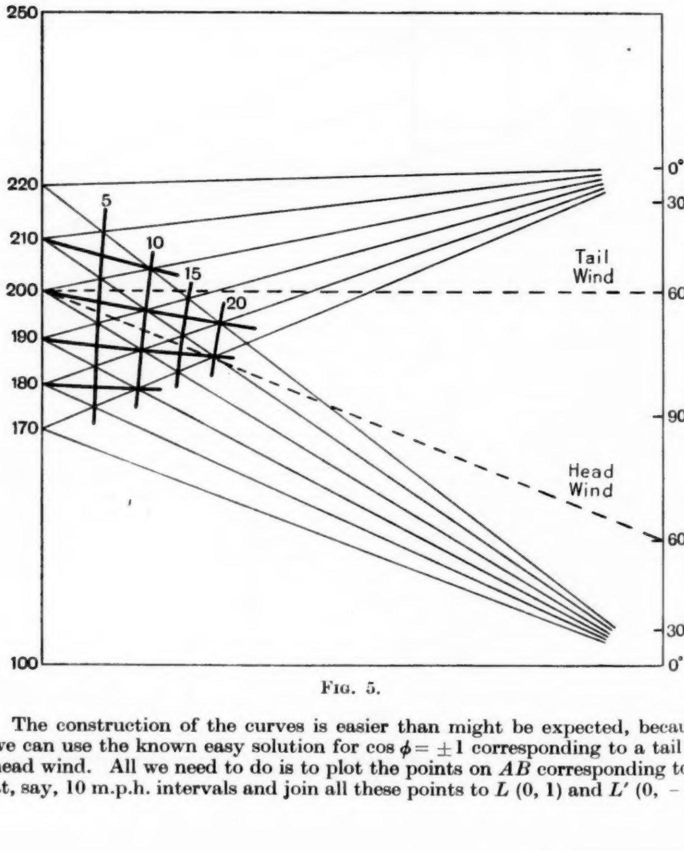


FIG. 5.

The construction of the curves is easier than might be expected, because we can use the known easy solution for $\cos\phi=\pm 1$ corresponding to a tail or head wind. All we need to do is to plot the points on AB corresponding to a at, say, 10 m.p.h. intervals and join all these points to L (0, 1) and L' (0, -1).

These lines cross each other, forming a series of quadrilaterals, opposite corners of which are joined up to form smooth curves. Part of the construction is shown in Fig. 5, which also gives the scale as actually constructed, and the solutions of two problems in dotted lines.

This determines most points except those at the top and bottom of the figure, and a certain amount of easy calculation is needed for these. The figure is cramped at the bottom for high wind speeds (but one always tries to avoid flying slow light aeroplanes in strong winds!), and does not fulfil my requirement of not having to use a ruler. I should like to see one which only involves following the lines of the squared paper with the finger. A transparent ruler is the best to use here, and the obvious one is the edge of the Douglas protractor itself.

Of course, the first nomogram can be used to find the ground speed. Once the drift has been found it may be added to, or subtracted from, ϕ and the result used as a new ϕ, which combined with the known drift and wind will lead back to a new value of a which is now the ground speed. This is very awkward and inaccurate on the ground and hopeless in the air.

The diagrams indicate the solution of this problem : Track required 025°, wind 15 m.p.h. from 325°, air speed 200 m.p.h. This gives $\phi=60^\circ$ (head wind), drift 4°, ground speed 192 m.p.h. Hence the course to steer is 021°. We find ϕ mentally by subtracting track and wind direction, the smaller from the larger, and either subtracting 180° from the result, or subtracting the result from 180° or 360°, so as to make $\phi \leqslant 90^\circ$. We can see from the map which side of the track the wind is and whether it is a head or a tail wind.

If the wind direction had been 265°, and everything else the same, then we should again have $\phi=60^\circ$, the same drift, but now a tail wind, and the ground speed would now be 207 m.p.h.

Interpolation between curves in both figures is easy, because it will not be necessary between wind curves. The weather people only give the wind speed to the nearest 5 m.p.h.

J. C. C.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

GLEANINGS FAR AND NEAR.

1627. The number rises at a regular rate, which is defined by the following mathematical equation : $N=7{\cdot}53A^{\cdot 471}$. N stands for the number of types, and A for the age. This equation tells us that the number of phoneme types equals 7·53 times the age raised to ·471 power, and that a curve plotted from this equation will be a parabola.—*Scientific American*, September 1949. [Per Mr. P. D. Greenall.]

ON THE MULTIPLICATION OF SERIES.

BY R. L. GOODSTEIN.

THOUGH the familiar proofs of the classical theorems on the multiplication of series are intuitively evident, they present, on reflection, many tiresome details, and by their piece-meal character tend to conceal the underlying unity of the results.

In the following account the multiplication theorems are all presented as simple consequences of a central formula in the theory of limits.

Notation. We denote by a_n a sequence of *positive* real numbers satisfying the condition $a_n/(a_0+a_1+\ldots+a_n)\to 0$; $A_n=\sum_0^n a_r$, so that $A_{n-1}/A_n\to 1$; s_n and t_n are any sequences, real or complex, and

$$s_n=s_n'+is_n'',\quad t_n=t_n'+it_n'',$$

$$U_n=\sum_0^n u_r,\quad V_n=\sum_0^n v_r,\quad w_n=\sum_0^n u_r v_{n-r}\quad\text{and}\quad W_n=\sum_0^n w_r.$$

1. THEOREM 1. *If $s_n\to s$ then*

$$\frac{a_0s_n+a_1s_{n-1}+a_2s_{n-2}+\ldots+a_ns_0}{a_0+a_1+a_2+\ldots+a_n}\to s.$$

1*. *The converse of Theorem 1 holds if s_n' and s_n'' are both non-negative and non-decreasing.*

Proof of Theorem 1. We observe first that for any fixed positive integer ν, and $n>\nu$,

$$\frac{A_{n-\nu}}{A_n}=\frac{A_{n-1}}{A_n}\cdot\frac{A_{n-2}}{A_{n-1}}\cdot\ldots\cdot\frac{A_{n-\nu}}{A_{n-\nu+1}}\to 1.$$

Write $z_n=s_n-s$ and choose $\nu=\nu_k$ so that $|z_n|<1/k$ for $n\geqslant\nu_k$; let

$$M=\max\{|z_r|,1\},\quad 0\leqslant r\leqslant\nu_1,$$

and let $n_k\,(>\nu_k)$ be such that, for $n\geqslant n_k$, $1-(A_{n-\nu}/A_n)<1/Mk$.

Then $$\left|\frac{a_0z_n+a_1z_{n-1}+\ldots+a_nz_0}{a_0+a_1+\ldots+a_n}\right|\leqslant\frac{1}{k}\frac{A_{n-\nu}}{A_n}+M\frac{A_n-A_{n-\nu}}{A_n}<\frac{2}{k},$$

for $n\geqslant n_k$, so that

$$\frac{a_0z_n+a_1z_{n-1}+\ldots+a_nz_0}{a_0+a_1+\ldots+a_n}\to 0.$$

Adding s to both sides, Theorem 1 follows.

Proof of 1*. It suffices to consider the case when s_n is real, non-negative and non-decreasing, since the general case follows by taking s_n to be s_n' and s_n'' in turn.

We have, for any positive integer k and $n>k$,

$$\frac{a_0s_n+a_1s_{n-1}+\ldots+a_ns_0}{a_0+a_1+\ldots+a_n}\geqslant s_k\frac{A_{n-k}}{A_n}+s_0\left\{\frac{A_n-A_{n-k}}{A_n}\right\}\to s_k,$$

and so $s\geqslant s_k$; write $d_n=s-s_n$ so that d_n is non-negative and non-increasing, then

$$0\leqslant d_n\leqslant\frac{a_0d_n+a_1d_{n-1}+\ldots+a_nd_0}{a_0+a_1+\ldots+a_n}=s-\frac{a_0s_n+a_1s_{n-1}+\ldots+a_ns_0}{a_0+a_1+\ldots+a_n}\to 0,$$

so that $d_n\to 0$, and so $s_n\to s$.

1·01. If $u_n \geqslant 0$ and $U_n \to U$ so that $1 - U_{n-1}/U_n = u_n/U_n \to 0$, then, by Theorem 1, if $s_n \to s$,

$$(u_0 s_n + u_1 s_{n-1} + \ldots + u_n s_0)/U_n \to s,$$

i.e.

$$u_0 s_n + u_1 s_{n-1} + \ldots + u_n s_0 \to Us.$$

1·1. Take $a_n = 1$, for all n, in Theorem 1, then, if $s_n \to s$,

$$(s_0 + s_1 + \ldots + s_n)/(n+1) \to s.$$

1·2. If $u_n \geqslant 0$, $u_n \to u$ and $s_n \to s$, then

$$(u_0 s_n + u_1 s_{n-1} + \ldots + u_n s_0)/(n+1) \to us.$$

For if $u \neq 0$, then

$$1 - U_{n-1}/U_n = u_n/U_n = \frac{u_n}{n+1}\Big/\frac{U_n}{n+1} \to 0, \text{ by } 1{\cdot}1,$$

and if $u = 0$ then $u_n/U_n < u_n/u_0 \to 0$, whence

$$\frac{u_0 s_n + u_1 s_{n-1} + \ldots + u_n s_0}{n+1} = \frac{u_0 s_n + u_1 s_{n-1} + \ldots + u_n s_0}{U_n} \cdot \frac{U_n}{n+1} \to us, \text{ by 1 and } 1{\cdot}1.$$

1·3. If $s_n \to s$ and $t_n \to t$, then

$$(s_0 t_n + s_1 t_{n-1} + \ldots + s_n t_0)/(n+1) \to st.$$

Since t_n converges, therefore t'_n, t''_n are convergent to t', t'' say, and bounded by T', T'' say. Then, by 1·2, $\left\{\sum_0^n (T' - t'_r) s_{n-r}\right\}\Big/(n+1) \to (T' - t')s$, and, by 1·1, $\left\{\sum_0^n s_{n-r}\right\}\Big/(n+1) \to s$, whence

$$\left\{\sum_0^n t'_r s_{n-r}\right\}\Big/(n+1) \to t's.$$

Similarly $$\left\{\sum_0^n t''_r s_{n-r}\right\}\Big/(n+1) \to t''s, \text{ and } 1{\cdot}3 \text{ follows.}$$

1·4. If $U_n \to U$ and $V_n \to V$, then

$$\frac{W_0 + W_1 + \ldots + W_n}{n+1} \to UV.$$

We prove first that if $h_n = \sum_0^n f_r g_{n-r}$ and $G_n = \sum_0^n g_r$, then $\sum_0^n h_r = \sum_0^n f_r G_{n-r}$. For $h_0 = f_0 G_0$, and if for some m, $\sum_0^m h_r = \sum_0^m f_r G_{m-r}$, then

$$\begin{aligned}\sum_0^{m+1} f_r G_{m+1-r} &= \sum_0^m f_r G_{m+1-r} + f_{m+1} g_0 \\ &= \sum_0^m f_r (G_{m-r} + g_{m+1-r}) + f_{m+1} g_0 = \sum_0^m h_r + h_{m+1} = \sum_0^{m+1} h_r.\end{aligned}$$

It follows that $W_n = \sum_0^n u_r V_{n-r}$ and *hence* that

$$\begin{aligned}&(W_0 + W_1 + \ldots + W_n)/(n+1) \\ &\qquad = (U_0 V_n + U_1 V_{n-1} + U_2 V_{n-2} + \ldots + U_n V_0)/(n+1) \to UV,\end{aligned}$$

by 1·3.

THEOREM 2. *If U_n, V_n and W_n are all three* convergent, *with limits U, V and W respectively, then*

$$W = UV.$$

For by 1·4, $\quad (W_0 + W_1 + \ldots + W_n)/(n+1) \to UV$;

and by 1·1, $\quad (W_0 + W_1 + \ldots + W_n)/(n+1) \to W$.

THEOREM 3. *If Σu_n is* absolutely convergent, *and Σv_n* converges conditionally, *with limits U and V respectively, then W_n* converges *to UV*.

For $\Sigma \mid u_n \mid$ converges and $\mid V_n - V \mid \to 0$, and so by 1·01

$$\mid W_n - (u_0 + u_1 + \ldots + u_n) V \mid$$
$$\leqslant \mid u_0 \mid \mid V_n - V \mid + \mid u_1 \mid \mid V_{n-1} - V \mid + \ldots + \mid u_n \mid \mid V_0 - V \mid \to 0,$$

whence $W_n \to UV$.

THEOREM 4. *If Σu_n and Σv_n are* absolutely convergent, *with limits U and V respectively, then Σw_n is* absolutely convergent, *with limit UV*.

(This includes the case of the product of two positive real series.)

By Theorem 3, Σw_n converges with limit UV. It remains to prove that Σw_n is *absolutely convergent*.

Let $w_n^* = \sum\limits_0^n \mid u_r \mid \mid v_{n-r} \mid$ and $W_n^* = \sum\limits_0^n w_r^*$, so that W_n^* is positive and monotonic increasing. By 1·4 we infer the convergence of

$$(W_0^* + W_1^* + \ldots + W_n^*)/(n+1),$$

whence by 1*, W_n^* converges; but $\mid w_n \mid \leqslant w_n^*$, and so Σw_n is absolutely convergent. R. L. G.

1628. *A true story.*

My baby son is exactly a fortnight old. Last night I leant over his cot and in a low voice told him the factors of $a^2 - b^2$. His lips at once parted in a smile. This was his very first smile, and the nurse confirmed that it really *was* a smile. [Letter from Mr. N. J. F. Craig.]

1629. For to explain how an actual 3-dimensional curve could come into existence through nothing else than the intersection of two impossible and 100 per cent non-existing bodies was quite beyond the giant (Quiribus Brown). As much indeed as it would have been for him to explain precisely how, through axis rotation in the field of vector analysis, Einstein's famous non-existent square root of minus 1 came into being.—H. S. Keeler, *The Murdered Mathematician*, pp. 125–6. [Per Mr. C. D. T. Owen.]

1630. People have bovarized themselves into the likeness of every kind of real or imaginary being. . . . What de Gaultier calls the bovaric angle between reality and assumed *persona* may be wide or narrow. In extreme cases the bovaric angle can be equal to two right angles. In other words, the real and assumed characters have exactly opposite tendencies. Most of us, I imagine, go through life with a bovaric angle of between forty-five and ninety degrees. —Aldous Huxley, *Writers and Readers*. (The words " bovaric " and " bovarized " are derived from " Madame Bovary.") [Per Mr. J. H. B. Smith.]

1631. One of the calculating machines . . . had completely automatic multiplication, *i.e.* the multiplier was put into the machine, a key was pressed, the multiplicand was then put into the machine, another key was pressed and the quotient (*sic*) appeared in the dials.—*The Times*, January 7, 1948, p. 2, col. 6. [Per Mr. I. FitzRoy Jones.]

SOME RESULTS IN THE THEORY OF MASS SYSTEMS.

A Study in "Calculus-dodging".

By H. M. Finucan.

Thin Uniform Rod.

1. Let the rod be of length l, line density d, mass $M = ld$. If I is its moment of inertia about one end, then from the theory of dimensions

$$I = CMl^2 = Cdl^3, \quad \text{.................................(1)}$$

where C is dimensionless, *i.e.* a constant, independent of M and l.

2. To find C, consider rods R^1, R^{11}, R^{111}, whose lengths and positions are as shown in Fig. 1; each is of line density d.

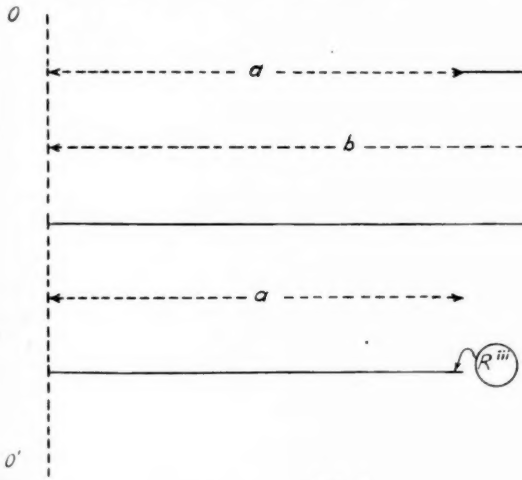


Fig. 1.

Then, considering M.I.'s about OO'

$$\begin{aligned} I^1 &= I^{11} - I^{111}\,; \quad \text{.................................(2)}\\ &= Cdb^3 - Cda^3 \text{ from (1)}\\ &= Cd(b-a)(b^2+ab+a^2)\\ &= CM^1(b^2+ab+a^2) \quad \text{.................................(3)}\end{aligned}$$

with an obvious notation.

Let b and a become nearly equal so that R^1 reduces to a small element distant a from OO' and

$$I^1 \doteqdot M^1a^2, \quad \text{.................................(4)}$$

but from (3) if $b \doteqdot a$:

$$I^1 \doteqdot CM^1 3a^2\,; \quad \text{.................................(5)}$$

and, comparing (4) and (5),

$$C = \tfrac{1}{3}$$

as required.

3. An alternative method of finding C after establishing, or assuming, (1)

is based on the theorem of parallel axes and makes no (implicit) use of limiting processes. In the following, the upper indices refer to the rods R^{iv} and R^{v} (Fig. 2), the lower indices to the axis AA' or OO' about which I is taken.

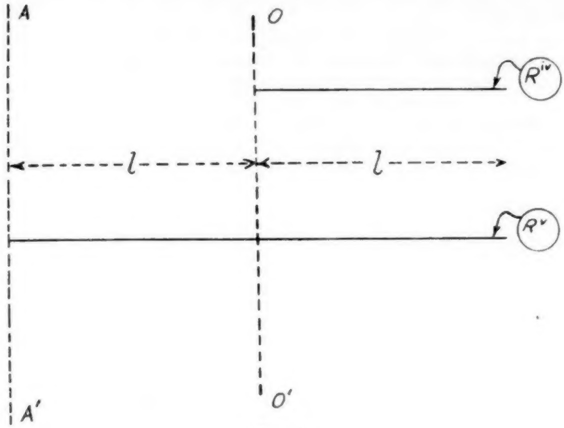


FIG. 2.

From (1) $$I_O^{\text{iv}} = CM^{\text{iv}}l^2 = Cdl^3, \quad\text{.......(6)}$$
also $$I_A^{\text{v}} = CM^{\text{v}}(2l)^2 = Cd8l^3\,; \quad\text{.......(7)}$$
then, from (6) and symmetry,
$$I_O^{\text{v}} = 2Cdl^3\,; \quad\text{.......(8)}$$
and from (8) and the theorem of parallel axes,
$$\begin{aligned} I_A^{\text{v}} &= 2Cdl^3 + (2ld)(l^2) \\ &= (2C+2)dl^3. \quad\text{.......(9)} \end{aligned}$$
Comparing (7) and (9),
$$8C = 2C + 2, \quad \text{whence } C = \tfrac{1}{3}.$$
Uniform Circular Wire.

4. To find the centroid of a circular arc of angle $2A$.

Methods of obtaining this result without integration are given in *Proc. Edin. Math. Soc.*, Vol. XVI (1898), pp. 49, 50; the following may also be of interest.

If a is the radius of the wire, then from the theory of dimensions,
$$OG = af(A).$$
To find $f(A)$ we regard the arc as a composite body formed of two equal arcs, each of semi-angle $A/2$ whose centroids G_1, G_2 are as shown in Fig. 3, and
$$OG_1 = OG_2 = af\left(\frac{A}{2}\right)\cdot \quad\text{.......(10)}$$
Since G is the midpoint of G_1G_2,
$$OG = OG_1 \cos\frac{A}{2},$$

$$f(A)=f\left(\frac{A}{2}\right)\cos\frac{A}{2}\,; \qquad \text{.............................(11)}$$

similarly,
$$f\left(\frac{A}{2}\right)=f\left(\frac{A}{4}\right)\cos\frac{A}{4}\,;$$

therefore
$$f(A)=f\left(\frac{A}{4}\right)\cos\frac{A}{2}\cos\frac{A}{4}\,,$$

and by repeated use of (11),

$$f(A)=f\left(\frac{A}{2^n}\right)\cos\frac{A}{2}\cos\frac{A}{4}\cos\frac{A}{8}\ldots\cos\frac{A}{2^n}\,.$$

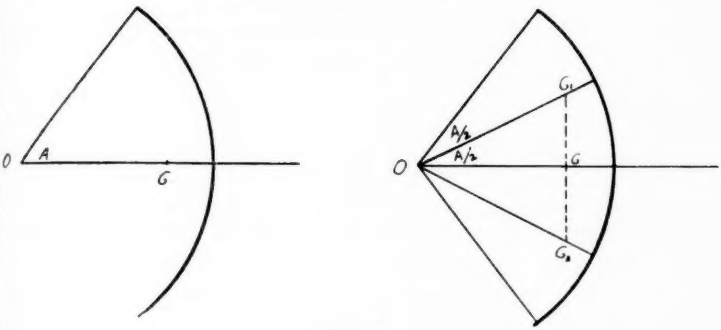


Fig. 3.

Now as n increases the arc of semi-angle $\dfrac{A}{2^n}$ becomes a small element, so that

$$f\left(\frac{A}{2^n}\right)\to 1,$$

and
$$\cos\frac{A}{2}\cos\frac{A}{4}\cos\frac{A}{8}\ldots\cos\frac{A}{2^n}\to\frac{\sin A}{A} \quad \text{(Euler)},$$

whence $f(A)=\dfrac{\sin A}{A}$, as required.

5. The M.I. of a uniform circular wire about a diameter is normally found without integration. The following method is therefore no shorter or simpler than the standard method, but is of interest in connection with para. 9 below.

Consider the quadrant AB (Fig. 4) of mass $\frac{1}{4}M$. Divide it into an even number of small equal arcs, each of mass m. Take a pair of such arcs P, Q, equidistant from OC (the bisecting radius); let $YOP=\alpha=XOQ$, then I denoting M.I. about YOY':

$$\begin{aligned} I \text{ of pair } P,\ Q &= ma^2\sin^2\alpha+ma^2\cos^2\alpha \\ &= ma^2 \\ &= 2m\,.\,OK^2, \text{ where } OK=\frac{a}{\sqrt{2}}, \qquad \text{................(12)} \end{aligned}$$

which is the same as I of two particles each of mass m lying together at K.

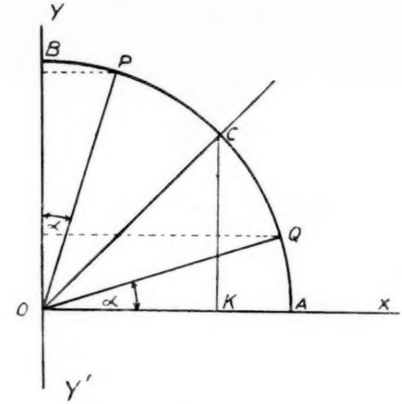


FIG. 4.

Thus every pair of arcs can be replaced by the mass of the pair at K, *i.e.* the whole quadrant may be replaced by its total mass at K without affecting I; thus

$$I \text{ of quadrantal wire} = \tfrac{1}{4}M\,.\,OK^2,$$

by symmetry $\quad I$ of circular wire $= M\,.\,OK^2 = \tfrac{1}{2}Ma^2.$ ....(13)

Uniform Circular Lamina.

6. The method of para. 2 above can be applied to find the M.I. of this body about an axis through its centre perpendicular to its plane. We consider the

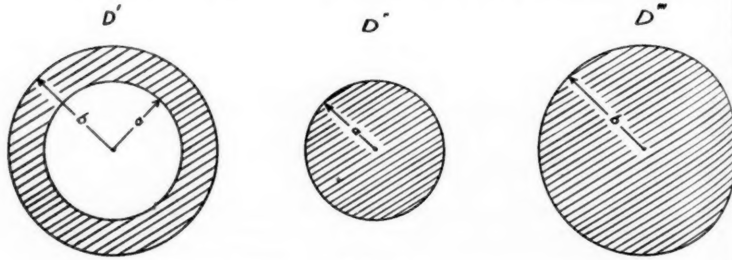


FIG. 5.

bodies D^1, D^{11}, D^{111} (Fig. 5) of surface density d. As in para. 2, no lower subscript is needed.

$$I = I^{111} - I^{11}$$
$$= kM^{111}b^2 - kM^{11}a^2$$

(from the theory of dimensions ; k is a constant to be determined)

$$= kd\pi b^4 - kd\pi a^4$$
$$= kd\pi(b^2 - a^2)(b^2 + a^2)$$
$$= kM^1(b^2 + a^2). \quad \text{..............................(14)}$$

And if $b \to a$,

$$I^1 \to M^1 a^2 \text{ (thin uniform wire) ; } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(15)}$$

also from (14), $$I^1 \to 2kM^1 a^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(16)}$$

Comparing (15) and (16),

$$k = \tfrac{1}{2} \text{ as required, } i.e.\ I^{11} = \tfrac{1}{2} M^{11} a^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(17)}$$

7. An alternative method of finding this M.I. throws some light on the similarity in form between (13) and (17). If the disc is of radius a we first divide a line of length a into small segments as follows. Construct a 90° arc $OACB$, centre O, $BOC = COA = 45°$ (Fig. 6).

Divide the arc AB into an even number of small equal arcs of length l and through the points of division draw perpendiculars to OA ; OA is thus

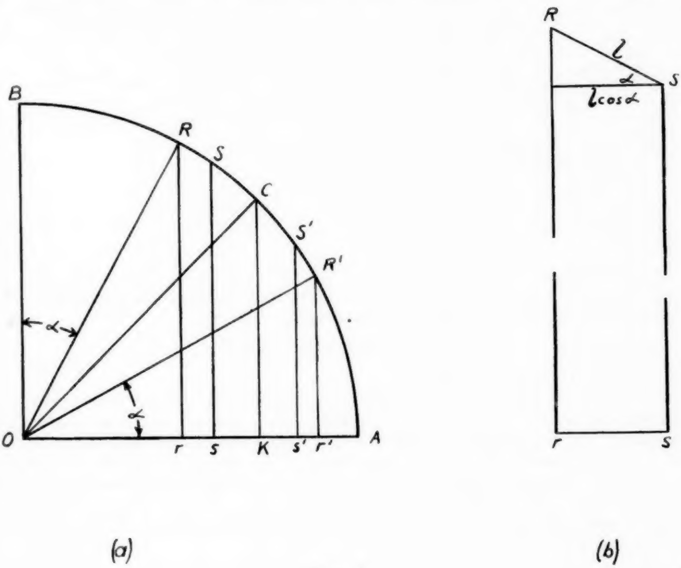


FIG. 6.

divided into a number of unequal segments, the arc RS ($BOR = \alpha$) generating a segment rs of length $l \cos \alpha$ (Fig. 6b) at a distance $a \sin \alpha$ from O ; the arc $R'S'$ ($AOR' = \alpha$) generating a segment $r's'$ of length $l \sin \alpha$ at a distance $a \cos \alpha$ from O. Two segments so related we shall call a " pair ".

8. Let a radius of the given disc, divided by the above method, rotate ; the points of division then trace out concentric circles which divide the disc into a number of thin rings associated in pairs (Fig. 7a). The ring " rs " of radius $a \sin \alpha$ and width $l \cos \alpha$ is of area $2\pi a \sin \alpha l \cos \alpha$; its mass is

$$2\pi a d \sin \alpha l \cos \alpha = \phi(\alpha). \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(18)}$$

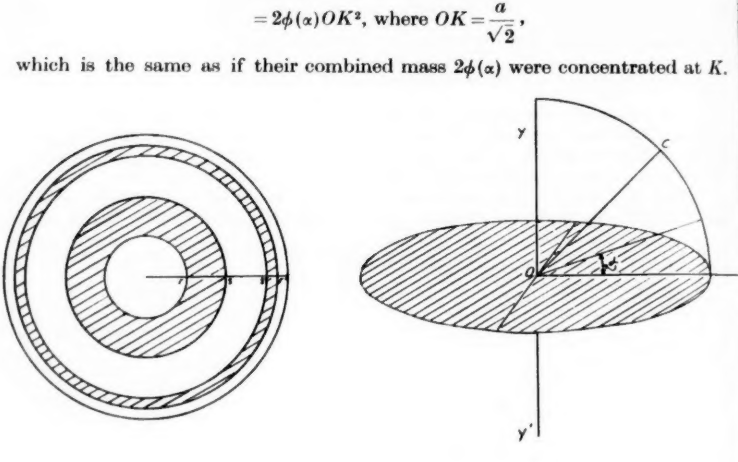
The mass of the ring " $r's'$ " is also $\phi(\alpha)$.

Then $\quad I_O$ of this pair of rings

$$= \phi(\alpha)a^2 \sin^2 \alpha + \phi(\alpha)a^2 \cos^2 \alpha$$

$$= 2\phi(\alpha)OK^2, \text{ where } OK = \frac{a}{\sqrt{2}},$$

which is the same as if their combined mass $2\phi(\alpha)$ were concentrated at K.



(b)

FIG. 7.

As the same is true of every pair, the complete disc may be replaced by its mass concentrated at K without changing I_O. Thus

$$I_O \text{ of disc} = M \,.\, OK^2 = \tfrac{1}{2}Ma^2. \qquad \text{.........................(19)}$$

9. It will be observed that

(i) For a non-uniform quadrantal arc, radius a, whose line density is symmetrical about C (Fig. 4), the result $I_{YY'} = \frac{1}{4}M\frac{1}{2}a^2$ (para. 5) can still be established by the method of para. 5 ($\frac{1}{4}M$ being the total mass of the arc).

(ii) A quadrantal arc whose line density varies as $\sin 2\alpha$ (Fig. 4) and a disc of equal radius a situated as in Fig. 7*b* are similar distributions of matter w.r.t. the axis YY'. This is clear from (18).

(iii) A uniform quadrantal arc and a uniform disc are not similar distributions w.r.t. YY',

Uniform Solid Sphere.

10. The M.I. of this body about a diameter can be found by the method of paras. 2, 6. Consider a spherical shell S^1 of radii a (inner) and b (outer), a sphere S^{11} of radius a, and a sphere S^{111} of radius b, all being of density d.

Then $\quad I^1 = I^{111} - I^{11}$

$$= M^{111}Cb^2 - M^{11}Ca^2, \text{ where } C \text{ is to be determined}$$

$$= \tfrac{4}{3}\pi Cd(b^5 - a^5)$$

$$= CM^1 \frac{b^5 - a^5}{b^3 - a^3}$$

$$= CM^1 \frac{b^4 + b^3a + b^2a^2 + ba^3 + a^4}{b^2 + ab + a^2}. \qquad \text{.............................(20)}$$

As b approaches a we have approximately :

$$I^1 = \tfrac{5}{3} C M^1 a^2 \text{ from (20)},$$

but also

$$I^1 = \tfrac{2}{3} M^1 a^2 \text{ (uniform spherical shell).}$$

Comparing these, $C = \tfrac{2}{5}$ as required.

Other Results.

11. The method of paras. 2, 6 and 10 is applicable to many similar problems —the evaluation of certain integrals being reduced to the determination of constants after the functional form has been established by the theory of dimensions. The reader will easily construct proofs of the formulae for the volume of a cone, the M.I. of a cone (solid or hollow) about its axis, etc., etc.

12. As a further example the position of the centroid of a solid cone will be found. Considering cones of fixed semi-angle, vertex O, the distance from O to the centroid of a cone of height h is kh where k is a constant independent of h. Then the centroid of a frustum of the cone from $h = b$ to $h = a$ is distant c from O where

$$c = \frac{a^3 \,.\, ka - b^3 \,.\, kb}{a^3 - b^3},$$

since the " original " and " removed " masses are in the ratio $a^3 : b^3$, *i.e.*

$$c = k \,.\, \frac{a^3 + a^2 b + ab^2 + b^3}{a^2 + ab + b^2}.$$

Since $a > c > b$, we have $k \to \tfrac{3}{4}$ as $a \to b$.

13. To show how this type of argument may be concluded with greater rigour we proceed as follows :

Let $k = \tfrac{3}{4} + z$ where z is a constant independent of a, b. Assuming that $a > b$, it is obvious dynamically that $a > c$, *i.e.*

$$a > (\tfrac{3}{4} + z)\,\frac{a^3 + a^2 b + ab^2 + b^3}{a^2 + ab + b^2};$$

simplifying we obtain

$$(a - b)(a^2 + 2ab + 3b^2) > 4z \,.\, 4b^3,$$

i.e.

$$(a - b)\,6a^2 > z \,.\, 16b^3, \quad \text{.................................(21)}$$

and the constant z must satisfy this for all values of a and b. But any fixed positive value of z fails to satisfy (21) if $(a - b)$ is taken small enough, or if, alternatively, $(a - b)$ being fixed, a and b are taken large enough. Therefore z cannot be positive.

In the same way, beginning with the relation $b < c$, it may be shown that z cannot be negative.

Hence $z = 0$, so that $k = \tfrac{3}{4}$ as required.

Uniform Triangular Lamina.

14. To find I for this body, about BC and parallel lines, let G be the centroid, E, F the mid-points of CA, AB ; draw AZ parallel to BC, and let lines through E and F perpendicular to BC meet BC at L, M and AZ at L', M' ; let GX be parallel to BC. We leave the reader to draw the figure.

We first find I_{EF}. If the triangles FMB, ELC are moved to positions $FM'A$, $EL'A$ respectively, I_{EF} is unaltered and the triangle becomes a rectangle of height h. Thus

$$I_{EF} = \tfrac{1}{12} M h^2;$$

and so by the theorem of parallel axes,

$$I_{GX} = \tfrac{1}{18} M h^2, \quad I_{BC} = \tfrac{1}{6} M h^2, \quad I_{AZ} = \tfrac{1}{2} M h^2.$$

H. M. F.

THE FREEDOM OF LINKAGES.

By R. H. Macmillan.

A linkage is an assembly of coupled bodies or links whose freedom of movement is restricted, after the fixture of one link, by the constraint imposed by their couplings. Frameworks and mechanisms are particular types of linkage. In this article the freedom on a plane or in space of linkages of an unlimited complexity is investigated by a general method whose application can readily be extended to space of four or more dimensions. The results obtained permit the analysis of any given linkage by mere substitution in the appropriate formula; but although the results are primarily intended to assist the analysis of existing linkages, it is shown how they can be used to a limited extent in the synthesis of new linkages to meet given requirements.

It is believed that formulae of such generality as those here derived have not been previously published in English, but the titles of certain German and Italian papers, unobtainable in this country, suggest that they may deal with the same subject, for which the name " number synthesis " appears to have been coined.(1)

1. *Definitions and nomenclature.*

The number of degrees of freedom possessed by a linkage is the number of independent parameters necessary completely to determine its configuration; it will be denoted by f_k where k is the number of dimensions in which the linkage is placed. When $f_k=0$, the linkage is rigid and forms a simply stiff framework. When $f_k=1$, the linkage is said to form a mechanism; in this case, if $k=3$, every point of a link is constrained to move on a fixed line and when $f_k=2$ every point moves on a fixed surface. Mechanisms having more than one degree of freedom are commonly used as differentials. If $f_k=-r$, the linkage is in general rigid and possesses r redundant constraints, but if it then behaves as a mechanism it has $(r+1)$ hidden constraints; this is considered in greater detail later.

The links are rigid members which maintain relatively fixed the position of the two or more couplings each of which connects it to one or more other links. The symbol n_p is used for the number of links having p couplings, so that the total number of links in the whole system is given by

$$n=n_2+n_3+n_4+\ldots=\sum_{p=2}^{\infty} n_p. \qquad (1)$$

Three general types of coupling are considered: those in a plane which permit a single degree of freedom between the members joined (that is, the relative position of the links is defined by a single parameter) such as the turning joint or hinge, and the sliding joint. The symbol j_p denotes the number of joints which connect p links in the plane.

In space, two types of joint are considered: the screw joint, which permits a single degree of freedom and includes as special cases the hinge, which is a screw of zero pitch, and the slide, which is a screw of infinite pitch. s_p is the number of screw joints, each connecting p members. The second type of joint in space is the ball and socket joint which permits three degrees of relative freedom between the links joined. The symbol j_p is again used for the number of such joints connecting p links. If the symbol c_p is introduced to cover the number of couplings connecting p links at each, no matter what the constraint they impose, then $c_p=j_p+s_p$ and the total number of couplings in a linkage is given by

$$c=c_2+c_3+c_4+\ldots=\sum_{p=2}^{\infty} c_p. \qquad (2)$$

From geometrical considerations the relation

$$2n_2 + 3n_3 + 4n_4 + \ldots = 2c_2 + 3c_3 + 4c_4 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or

$$\sum_{p=2}^{\infty} pn_p = \sum_{p=2}^{\infty} pc_p$$

evidently holds for all linkages, in space of any number of dimensions.

The joints denoted by the letter j are normally called pin joints ; couplings, whether in a plane or space linkage, which connect two members only will be described as *simple joints* and, similarly, links connecting two pin joints only will be called *simple links*. If a closed chain consists entirely of simple joints and simple links it constitutes a *simple chain*, for which it follows immediately from (3) that $n = c$. A closed chain in which the members are coupled exclusively by sliding joints is called a *prismatic chain*. The number of independent prismatic chains in a linkage is denoted by q. Members connected by hinges and sliding joints are said to form *lower pairs*. The type of contact which occurs between a cam and its follower is called *higher pairing*. The number of *higher pairs* in a linkage will be denoted by h. Prismatic chains and higher pairs in space will not be discussed, as they can be treated easily by analogy with their treatment in plane linkages.

2. *Theory of plane linkages.*

The number of independent parameters required to define the position of a link in a plane is three. These might be, for example, the two co-ordinates of a specified point together with the inclination of a line fixed in the link. Alternatively, the position of two points (x_1, y_1) and (x_2, y_2) might be specified, but these are not independent since $\sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$ is constant, being equal to the distance between the points. The number of parameters required completely to define the position in a plane is therefore $3n$.

Since a simple joint permits only one degree of relative freedom to remain out of three it must impose two constraints ; this is also evident in the case of a hinge from the fact that the position of one point in one link is fixed relatively to one point in the other. Adding a third link to the joint imposes two more constraints, and similarly for a fourth link, so that in general a joint connecting p members imposes $2(p-1)$ constraints. The number of constraints imposed by all the joints of a linkage is accordingly

$$2j_2 + 4j_3 + 6j_4 + \ldots = \sum_{2}^{\infty} 2(p-1)j_p.$$

The number of constraints imposed by the fixing of one link is three, and the number of degrees of freedom of the linkage is $3n$ less the constraints imposed, so that

$$3n - (2j_2 + 4j_3 + 6j_4 + \ldots) - 3 = f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or

$$3\Sigma n_p - 2\Sigma(p-1)j_p - 3 = f_2.$$

It is instructive to approach the problem from a different viewpoint. The configuration of the linkage can also be specified by the positions of the joints, which demand $2j$ parameters to define them. A simple link imposes one constraint, since it fixes the distance between two joints ; a link with three joints imposes three constraints, as it must define the triangle formed by the three joints ; and one with p joints imposes $(2p-3)$ constraints. Three constraints are necessary to fix one link, as before, so that

$$2j - (n_2 + 3n_3 + 5n_4 + \ldots) - 3 = f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

or

$$2\Sigma j_p - \Sigma(2p-3)n_p - 3 = f_2.$$

Subtracting (5) from (4) gives

$$(4n_2 + 6n_3 + 8n_4 + \dots) - (4j_2 + 6j_3 + 8j_4 + \dots) = 0,$$

which is equation (3), so that of equations (3), (4) and (5) only two are independent.

A number of important special cases can be immediately deduced from the equations derived above. If a linkage consists exclusively of simple links, from (5)

$$2j - n - 3 = f_2, \quad \dots\dots(6)$$

so that for a simply stiff pin-jointed frame with simple links

$$2j - n = 3. \quad \dots\dots(7)$$

If the links may be complex but the joints are all simple, from (4)

$$3n - 2j - 3 = f_2, \quad \dots\dots(8)$$

so for a mechanism with simple joints,

$$3n - 2j = 4. \quad \dots\dots(9)$$

Adding equations (4) and (5) and dividing by two gives

$$(n_2 - n_4 - 2n_5 - 3n_6 - \dots) - (j_3 + 2j_4 + 3j_5 + \dots) - 3 = f_2, \quad \dots\dots(10)$$

so that if the joints are all simple,

$$n_2 - (n_4 + 2n_5 + 3n_6 + \dots) - 3 = f_2, \quad \dots\dots(11)$$

which is independent of the number of joints. This equation shows that a mechanism must include at least four links with two elements. Using equation (1) the last equation becomes

$$n - (n_3 + 2n_4 + 3n_5 + \dots) - 3 = f_2. \quad \dots\dots(12)$$

If, on the other hand, no link has more than three joints, equation (10) gives

$$n_2 - (j_3 + 2j_4 + 3j_5 + \dots) - 3 = f_2, \quad \dots\dots(13)$$

where

$$n_2 = n - n_3.$$

Finally, if no joint connects more than three links and no link more than three joints, we have the remarkably simple relation

$$n_2 - j_3 - 3 = f_2. \quad \dots\dots(14)$$

If n_2 is eliminated between equations (4) and (5), we have

$$4(f_2 + 3) = (4j_2 + 2j_3 - 2j_5 - 4j_6 - \dots) - (6n_3 + 12n_4 + 18n_5 + \dots).$$

Then if all the links are simple,

$$f_2 + 3 = j_2 + \tfrac{1}{2}(j_3 - j_5 - 3j_7 - 5j_9 - \dots) - (j_6 + 2j_8 + \dots), \quad \dots\dots(15)$$

a relation independent of the number of links and indicating that

$$(j_3 - j_5 - 3j_7 - 5j_9 - \dots) \quad \text{and hence} \quad (j_3 + j_5 + j_7 + \dots)$$

must be always even, for $f_2 = 0$ or $f_2 = 1$.

3. *Higher pairs and prismatic chains.*

In the analysis of the last section it has been tacitly assumed that the linkages do not contain chains of simple links having only sliding joints, each of which links defines the angular relation between its two sliders. If a link slides on a fixed link it has one degree of freedom, and if a further link slides on this one it has two degrees of freedom, but if a third link slides on the second it gains no further freedom, for it still cannot rotate, and similarly

for any additional sliding links. For a closed chain of links to form a prismatic chain there must necessarily be a relation between the angles defined by the links of the chain, their sum being 2π, so that the number of independent parameters required to define the linkage is reduced by one. The freedom of a plane linkage containing q independent prismatic chains is accordingly (f_2+q). It is also evident that there must be at least two turning pairs in every chain, for a single one would be locked by the remaining sliders ; modifications of the formulae on the lines above must be introduced to account for such cases.

Figure 1 shows a plane linkage with all joints simple and $n=n_3=6$. From equation (11), $f_2=n_2-3=-3$. It is a triply redundant structure. If all the joints are sliding joints, however, it contains four independent chains, so $q=4$ and the freedom is now $f_2+q=1$. The linkage is a mechanism.

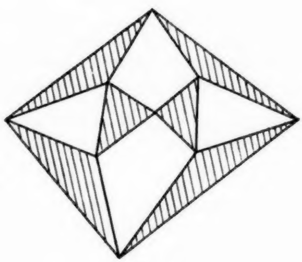


FIG. 1.

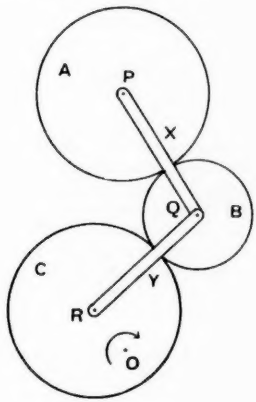


FIG. 2.

Higher pairing is a form of joint which causes only a single degree of constraint. Therefore, when a mechanism contains h such pairs, they should be counted as joints and the actual freedom of the linkage is then given by (f_2+h), for h fictitious constraints have thus been introduced. One of the commonest forms of higher pairing is the contact of gear teeth : Figure 2 represents a gear chain linkage used in paper manufacture. The gear C is eccentrically driven at the fixed point O. B is an idling wheel with centre Q maintained in position by the links PQ and QR. The motion is taken from the wheel A whose axis is fixed at P. There is higher pairing at X and Y. Here $n=6$, $j_2=4$, $j_3=2$, $h=2$, and from equation (4) the freedom is

$$f_2+h=3n-(2j_2+4j_3)-3+h=1,$$

indicating that the linkage is a mechanism.

4. *Exceptional freedom and redundancy.*

The formulae which have been derived are necessary but not sufficient conditions for the freedom of a linkage. Although application of a formula might suggest that a certain linkage is just rigid (*i.e.* $f_k=0$), it might happen that one part is free while another part contains redundant constraint. An example of this is shown in Figure 3, where the diagonal members should not

both be bracing the same rectangle. Such a state of affairs can only be avoided by ensuring that every closed chain in the linkage possesses the required constraint of freedom. With a space linkage such a process would become very arduous, if not impossible, and more refined methods must be used.[2]

Another possiblility is that when certain relations exist between the dimensions of the links, the freedom of some modes of motion may be effectively destroyed more than once, so that the linkage has greater real freedom than the apparent freedom indicated by the formula. In all such cases the difference between the real and the apparent freedom represents a number of hidden redundancies. Such redundancies, no less than those which are obvious, should be removed if possible, for they introduce the probability of self-strain into the linkage, as a result of manufacturing tolerances and differential thermal expansion; they also complicate the design of the linkage, since the loads taken by the links become statically indeterminate, being dependent on the elasticity of the members.

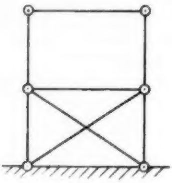


FIG. 3.

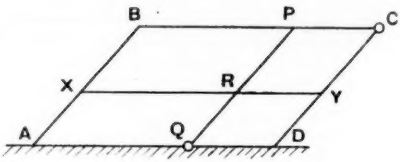


FIG. 4.

Consider, for example, the four bar parallelogram linkage $ABCD$ shown in Figure 4. All the joints are simple, so that from equation (11)

$$f_2 = n_2 - 3 = 4 - 3 = 1,$$

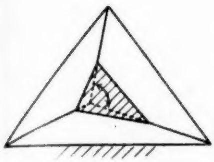
showing that the linkage forms a mechanism. If a further link XY is added between AB and CD so that $AX = DY$, the linkage still has one degree of freedom, but $n_2 = 3$, $n_3 = 2$, so $f_2 = n_2 - 3 = 0$, showing that the freedom is apparently zero. Suppose now a further bar PQR is added, with $BP = AQ$; the linkage is still free but now $n_2 = 0$, $n_3 = 6$, giving $f_2 = -3$, which indicates that there are four redundant constraints. The mechanism would behave more satisfactorily if two of the hinges, say those at Q and C, were removed, for then $n_2 = 4$, $n_3 = 2$, giving $f_2 = 4 - 3 = 1$.

A more elaborate plane mechanism possessing exceptional freedom is the Double Four,[3] which has $n_2 = 0$, $n_3 = 8$, $j_2 = 12$, $j_3 = 0$, with special dimensions for the links. As the joints are all simple, equation (8) shows that $f_2 = 24 - 24 - 3 = -3$, or from equation (14) $f_2 = n_2 - j_3 - 3 = -3$. There are therefore four degrees of redundancy.

Another type of exceptional freedom is particularly important in the engineering field. In this case an infinitesimal deformation of the first order in an apparently stiff frame may result from second order deformations of the links. Such linkages have all the disadvantages of redundant structures, for the loads in the members are statically indeterminate, even though $f_k = 0$. The whole question of exceptional freedom is beyond the scope of the present article, and it must suffice to record a few examples of the last type mentioned. The linkages shown in Figures 5 and 6 are both apparently rigid, yet possess infinitesimal freedom as indicated by the arrows. A final example is the cross-braced hexagon of Figure 7, in which $n_2 = 9$, $j_3 = 6$, giving

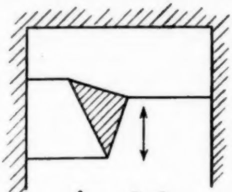
$$f_2 = 9 - 6 - 3 = 0,$$

yet small deformations are possible if the joints lie on a conic.[4] Such configurations, whose variety is infinite, are known as "critical forms".



$f_2 = n_2 - j_3 - 3 = 0$

FIG. 5.



$f_2 = n_2 - 3 = 0$

FIG. 6.

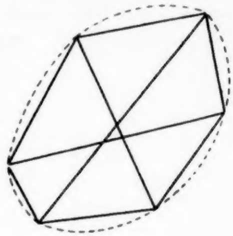


FIG. 7.

5. *Plane mechanisms and frameworks.*

Before proceeding to a consideration of linkages in space, it is interesting to consider the application of the formulae to some of the more complex linkages in actual use.

Figure 8 shows diagrammatically the plane linkage used in the Walschaert steam engine valve gear. Figure 9 is the same linkage reduced to skeleton form, the links being numbered as in the previous figure with 1 as the fixed link. Sliding joints are marked with an *S*. All the joints are simple and $n_2 = 6$, $n_3 = 5$, $n_5 = 1$, so equation (11) gives $f_2 = n_2 - 2n_5 - 3 = 1$.

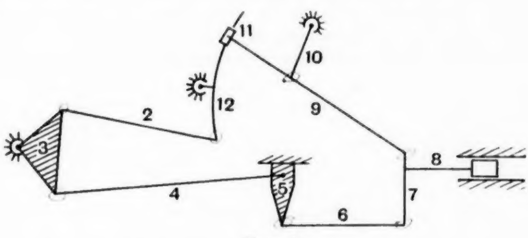


FIG. 8.

In Figure 10 is the skeleton of a rather more elaborate plane mechanism, the Brown Valve Gear. Neither the links nor the joints are all simple.

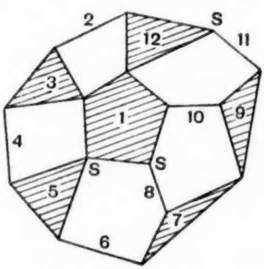


FIG. 9.

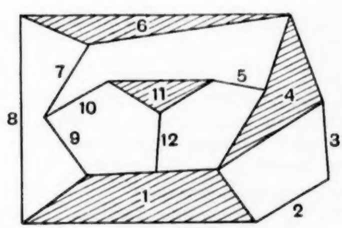


FIG. 10.

$$n=12\,;\quad n_2=8,\quad n_3=2,\quad n_4=1,\quad n_5=1.$$

$$j=15\,;\quad j_2=14,\quad j_3=1.$$

From equation (4) $\quad f_2=36-(28+4)-3=1,$

or from equation (5) $\quad f_2=30-(8+6+5+7)-3=1.$

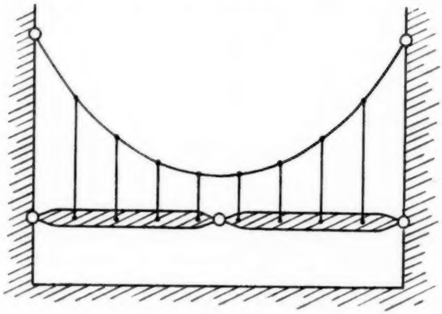


FIG. 11.

An interesting example of a fairly complex plane framework is the suspension bridge. Let there be v vertical ties and a central hinge in the roadway, which is also hinged to each abutment as in Figure 11. We have $n=2v+4$, $j_2=v+5$, $j_3=v$. Therefore $f_2=3(2v+4)-2(v+5)-4v-3=-1$, from equation (4). The bridge has a single redundancy, which could be eliminated by allowing a horizontal sliding motion to the road at one abutment.

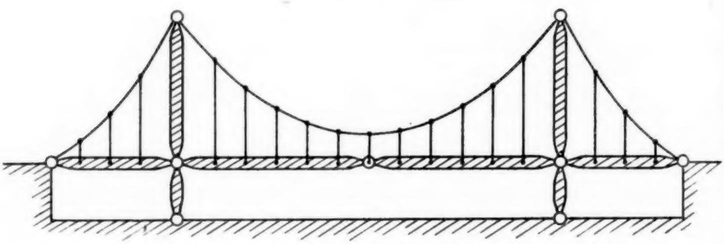


FIG. 12.

A more elaborate type of suspension bridge is shown in Figure 12. Here $n=2v+12$, $j_2=v+1$, $j_3=v+5$, $j_4=2$. Equation (4) gives

$$f_2=3(2v+12)-2(v+1)-4(v+5)-12-3=-1,$$

as for the simpler bridge. To allow for thermal expansion, the cable might be permitted to slide over the tops of the towers; in this case the towers would not be hinged at the roadway for then their upper parts would be unconstrained.

6. *Theory of space linkages.*

With suitable modifications the methods which have been applied in the plane can be extended to linkages in space. Six independent parameters

are needed to define completely the position of a body in space: three, for instance, might be used to define the position of any point in the body, two more to give the direction of a line fixed in the body and a further one to define the rotation of the body about this line. Alternatively, the orientation of the body might be described by means of Euler's three angular co-ordinates. Again, nine parameters might be used to define the vertices of a triangle fixed in the body, in conjunction with the three relations defined by the lengths of the sides of the triangle. $6n$ parameters are therefore required to define the position of the n members of a linkage in space.

Since a simple screw joint permits only one degree of relative freedom it must impose five constraints. If a third link is added at the screw, five more constraints are added and a screw joint connecting p links imposes $5(p-1)$ constraints. The fixing of one link imposes a further six constraints, so the freedom of the linkage is given by

$$f_3 = 6n - (5s_2 + 10s_3 + 15s_4 + \ldots) - 6 \quad \ldots\ldots(16)$$

or

$$f_3 = 6\Sigma n_p - 5\Sigma(p-1)s_p - 6.$$

Use of the alternative approach, in which the position of the screws is first defined, leads to

$$f_3 = 5s - (4n_2 + 9n_3 + 14n_4 + \ldots) - 6, \quad \ldots\ldots(17)$$

or

$$f_3 = 5\Sigma s_p - \Sigma(5p-6)n_p - 6.$$

Subtracting (16) from (17) leads to equation (3), so that of these three equations only two are independent.

Some special cases may be noted. If all the links are simple,

$$f_3 = 5s - 4n - 6; \quad \ldots\ldots(18)$$

and if all the joints are simple,

$$f_3 = 6n - 5s - 6. \quad \ldots\ldots(19)$$

If all the joints are simple, eliminating s_2 between (16) and (17) gives

$$2(f_3 + 6) = 2n_2 - 3n_3 - 8n_4 - 13n_5, \quad \ldots\ldots(20)$$

an expression for the freedom independent of the number of joints, and indicating that the sum of the odd links must be even in a mechanism or framework. If all the links are simple we may derive similarly

$$f_3 = s_2 - (s_3 + 3s_4 + 5s_5 + \ldots) - 6, \quad \ldots\ldots(21)$$

which is independent of the number of links.

Let us now consider the pin-jointed linkage in space. Arguments similar to those already used lead to

$$3j - (n_2 + 3n_3 + 6n_4 + 9n_5 + \ldots) - 6 = f_3 \quad \ldots\ldots(22)$$

or

$$3\Sigma j_p - 3\Sigma(p-2)n_p - n_2 - 6 = f_3.$$

Alternatively,

$$6n - n_2 - 3(j_2 + 2j_3 + 3j_4 + \ldots) - 6 = f_3 \quad \ldots\ldots(23)$$

or

$$6\Sigma n_p - n_2 - 3\Sigma(p-1)j_p - 6 = f_3.$$

As before, equation (3) can be deduced from (22) and (23). The obtrusion of the term n_2 in these equations results from the fact that a link connecting only two pin joints in space requires only five co-ordinates instead of six to

determine it, since rotation of the link about the line joining the joints is immaterial. If all the joints are simple,

$$6n - n_2 - 3j - 6 = f_3 \; ; \qquad \text{(24)}$$

and if all the links are simple,

$$3j - n - 6 = f_3. \qquad \text{(25)}$$

If all the links are simple, elimination of n_2 between (22) and (23) leads to

$$f_3 + 6 = 2j_1 + \tfrac{3}{2}j_3 + j_4 + \tfrac{1}{2}j_5 + \ldots, \qquad \text{(26)}$$

independent of the number of links and indicating that the sum of the odd links must be even. If all the joints are simple, elimination of j_2 between (22) and (23) gives

$$f_3 + 6 = 2n_1 + \tfrac{3}{2}(n_3 - n_5 - 3n_7 - \ldots), \qquad \text{(27)}$$

independent of the number of joints and showing that the sum of the odd joints must be even, in a space frame or mechanism.

Finally, we shall consider a space linkage in which the members may contain screw joints and pin joints. The relation found is

$$6n - n_{2j} - 3(j_2 + 2j_3 + 3j_4 + \ldots) - 5(s_2 + 2s_3 + 3s_4 + \ldots) - 6 = f_3, \qquad \text{(28)}$$

where n_{2j} is the number of links with two pin joints only. If all the joints and all the links are simple,

$$6n - n_{2j} - 3j - 5s - 6 = f_3. \qquad \text{(29)}$$

7. *Space mechanisms and frameworks.*

The simple screw kinematic chain in space has one degree of freedom and $n = s$. It follows immediately from (18) and (19) that $n = 7$. Suppose we wish to find the possible mechanisms which can be formed from simple links and links with three screw joints only, all joints being simple; then equation (20) gives $2n_2 - 3n_3 = 14$, the simplest solutions of which are $n_2 = 10$, $n_3 = 2$ and $n_2 = 13$, $n_3 = 4$; this is an example of the use which may be made of the formulae for kinematic synthesis.

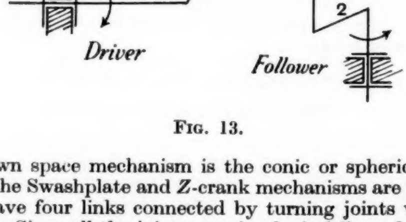


FIG. 13.

The best-known space mechanism is the conic or spheric chain, of which Hooke's Joint, the Swashplate and Z-crank mechanisms are practical applications. These have four links connected by turning joints whose hinge-lines meet in a point. Since all the joints are simple, it follows from equation (21) that $f_3 = -2$. Since the mechanism operates with a single degree of freedom there are three hidden redundant constraints. As such constraints are

undesirable for the reasons enumerated in the case of plane linkages, these redundancies are better eliminated; it is, in fact, common practice with Hooke's Joint to permit the axial movement of one shaft; it would also be permissible to allow sliding as well as turning at the two hinges of the intermediate member. Another four-bar space linkage possessing exceptional freedom, due to the presence of three redundant constraints, is the skew isogram discovered by G. T. Bennett in 1903.[5], [6] It has been shown [7] that these are the only four-bar space mechanisms, but other simple space mechanisms with less than seven links are known,[8] and it is unlikely that the list is yet exhausted.

A simple space chain which has found a number of practical applications, such as the control of a boat's rudder, is used for transmitting an oscillating motion between skew shafts. The arrangement of four links for perpendicular shafts is shown in Figure 13 diagrammatically; link 1 is here fixed, but the cross-member 3 is coupled to 2 and 4 by joints having two degrees of freedom, since they permit sliding as well as rotation. If these joints were converted to simple hinges it would be necessary, for the freedom of the mechanism, to permit sliding at points such as A and B, thereby introducing two new members and thus making a simple chain of six links in all. Since such a chain is in general rigid, there is evidently one redundancy. The reader may care to consider where this hidden constraint lies and how best to eliminate it.

Consider now eight triangular plates forming an octahedron hinged along its edges; this constitutes a screwed linkage in space, with $n=8$, $s=12$, and all the joints simple. Equation (19) gives immediately $f_3 = 48 - 60 - 6 = -18$. It is remarkable that under certain dimensional conditions, such an octahedron may be deformable with nineteen hidden redundancies.[9]

Another simple six-bar space mechanism is Sarrus' Parallel motion, shown in Figure 14. Link 1 is fixed and the hinges of links 2 and 3 are parallel, as are the hinges of links 5 and 6. The motion of any point on 4 is rectilinear.

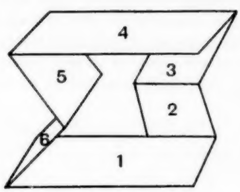


FIG. 14.

Discovered in 1853, this was the first exact "parallel motion" discovered. The redundant constraint could be removed by allowing relative rotation of the hinges of 4 about an axis normal to their plane.

An example of the application of the formulae for pin-jointed space linkages is afforded by the following problem. It is desired to find out how many diagonal members are required to brace the frame shown in Figure 15. The attachments at $ABCD$ and at $KLMN$ are to rigid bodies. The bracing is to be added between the existing joints. Then $f_3 = 0$, $j = 12$, $n_4 = 2$.

From equation (22), $3j - (n_2 + 6n_4) - 6 = f_3$.

Therefore $n_2 = 18$.

But there are twelve simple links already, so that six more are required. In adding these links it would be advisable to check the rigidity of each bay

separately, for although a certain degree of synthesis has been achieved, the formula does not indicate where the additional links should be placed.

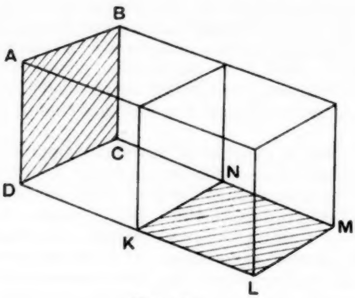


FIG. 15.

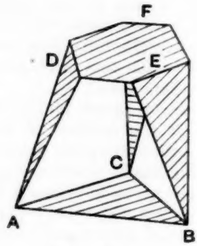


FIG. 16.

A common application of a space linkage involving both turning and pin joints is the instrument tripod, as used for a theodolite, for example. This is shown in Figure 16. The three legs are pinned to the ground at A, B and C, and hinged to the supporting plane along D, E and F.

$$n=5, \quad n_{2j}=0, \quad j_2=3, \quad s_2=3.$$

From equation (28), $\quad f_3 = 30 - 9 - 15 - 6 = 0.$

The structure is simply stiff, and affords an example of good "kinematic design".

8. *Conclusion.*

The use made of the formulae derived in this article has shown them to be readily applicable to a wide variety of structural and kinematic problems. It has been shown that the general equations reduce to very simple expressions in a number of important special cases, perhaps the most remarkable of which is that for a plane linkage, none of whose joints or links have more than three attachments :

$$f_2 = n_2 - j_3 - 3. \quad \text{.......(14)}$$

For a space linkage with simple screw joints,

$$f_3 = 6n - 5s - 6 \; ; \quad \text{.......(19)}$$

and for a space linkage with simple links,

$$f_3 = 3j - n - 6. \quad \text{.......(25)}$$

It must be emphasised that the freedom given by the formulae is the minimum value ; in certain critical forms, the actual freedom may be greater than that indicated, by an amount which must be the number of hidden redundancies. It is necessary also, of course, that the links should be so arranged that every set of joints which form a closed system possess exactly the correct number of constraints.

Investigation along the lines adopted can very easily be extended to space of more than three dimensions. In the case of a pin-jointed linkage in k dimensions, for instance, $\frac{1}{2}k(k+1)$ parameters are needed to define the position of the body, so that for simple links the relation is evidently

$$f_k = kj - n - \tfrac{1}{2}k(k+1). \quad \text{.......(30)}$$

The minimal simply stiff frames for $k=2$ to $k=5$ are shown diagrammatically in Figure 17.

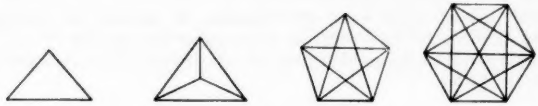


FIG. 17.

$k=2$	$k=3$	$k=4$	$k=5$
$j=3$	$j=4$	$j=5$	$j=6$
$n=3$	$n=6$	$n=10$	$n=15$

R. H. MACMILLAN.

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1632. CONVERSATION PIECE.

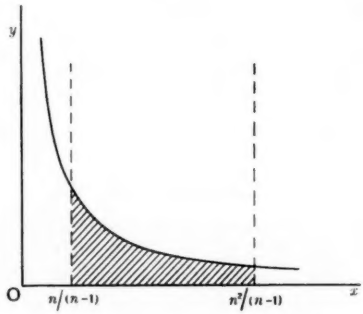
Der König : " Welche Wissenschaft haben Sie besonders studiert? "
Der Gelehrte : " Alle, Sire! "
Der König : " Sie sind also auch ein guter Mathematiker? "
Der Gelehrte : " Jawohl, Sire! "
Der König : " Wer hat Sie in Mathematik unterrichtet? "
Der Gelehrte : " Ich selbst."
Der König : " Dann sind Sie also ein zweiter Pascal? "
Der Gelehrte : " Jawohl, Sire."

So berichtet Thiébault über des erste Zusammentreffen das Mathematikers Johann Heinrich Lambert mit dem Hohenzollern Friedrich II in Potsdam, März 1764. " Le plus grand imbécile que j'ai jamais vu! " so fasste der Fürst seinen ersten Eindruck zusammen, und die beabsichtigte Anstellung an der Berliner Akademie schien zu entschwinden. Ganz anders dachte Lambert über seine Aussichten : " Wenn der König mich nicht behält, so wäre es ein Flecken in seiner Geschichte." Er Behielt recht.—*Elemente der Mathematik*, Bd. II, Nr. 3, May 1947, p. 70.

MATHEMATICAL NOTES.

2105. *The distribution of the primes.*

It is well known that $\pi(x)$, the number of primes not exceeding x, is asymptotic to $x/\log x$. It is difficult to visualise the growth of $\pi(x)$ from this, since a graph with a reasonable range of values can only be obtained by using a logarithmic scale.



In this connection the following observation may be of interest. With two adjacent edges of a thin plane uniform rectangular lamina as axes of x and y draw a curve $xy=k$. Let $f(n)$ be the abscissa of the centroid of that portion of the lamina bounded by the curve, the x-axis and the ordinates $x=n/(n-1)$, $x=n^2/(n-1)$. Then $\pi(n)$ is asymptotic to $f(n)$. In fact, integrating from $n(n-1)$ to $n^2/(n-1)$,

$$f(n)=\int k\,dx\Big/\int\frac{k\,dx}{x}=n/\log n\sim\pi(n).$$

1. Since the length along the x-axis between $n/(n-1)$ and $n^2/(n-1)$ is n, this length may be regarded as representing " all the numbers " not exceeding n, while $f(n)$ represents the primes not exceeding n. By drawing the curve and estimating by eye the position of the centroid, remembering that

$$n/(n-1)\to 1 \quad \text{and} \quad n^2/(n-1)\sim n$$

as $n\to\infty$, a fairly satisfactory conception of the growth of the number of primes may be obtained.

2. It is interesting to approximate to the area. For example, taking two rather extreme cases, a very weak approximation is obtained by considering it as a trapezium, when

$$f(n)\simeq\tfrac{1}{3}n(n^2+5n+3)/(n^2+n-2),$$

and a better by taking the area as made up of $n-1$ rectangles each of width $n/(n-1)$, when

$$f(n)\simeq\frac{n}{2(n-1)}+n\Big/\sum_{1}^{n-1}\frac{1}{r}\,.$$

3. Mr. P. Hall, of King's College, Cambridge, has suggested that a similar interpretation of $li(n)$, which gives a better approximation to $\pi(n)$ than $n/\log n$, would be of significance as well as of interest. This, however, seems to be a matter of some difficulty.

D. E. Mansfield.

2106. *Note on solution of triangles.*

Note No. 1966 brings up the question of the best method of teaching solution of triangles to the non-mathematician who has to use trigonometry in his everyday work. In particular, toolroom problems involve three-dimensional trigonometry, and some of the American methods in this class of problem have much to commend them. I give below the solution of the general triangle when the data are (i) 3 sides and (ii) 2 sides and the included angle. The other cases are covered by the sine rule and so do not offer any difficulty of treatment.

(i) Given a, b, c.

To find " x ".

$$AD^2 = c^2 - (\tfrac{1}{2}a + x)^2 = b^2 - (\tfrac{1}{2}a - x)^2.$$

Hence $\quad x = (c-b)(c+b)/2a.$

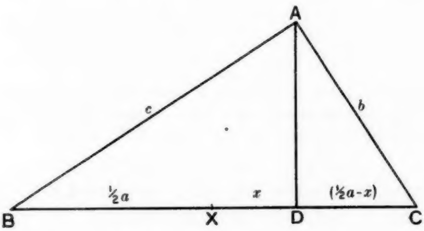
Then $\quad \cos B = (\tfrac{1}{2}a + x)/c, \quad \cos C = (\tfrac{1}{2}a - x)/c.$

(ii) Given a, b, C.

The Americans normally teach the six trigonometrical ratios, and so they invert the formula suggested in Note 1966, thus obtaining

$$\cot A = \frac{b}{a} \operatorname{cosec} C - \cot C,$$

a form which needs a little less calculation than the one given.



This, however, still avoids the question as to what we shall do if the side " c " is required. We have four possibilities open to us :

(*a*) we can use the above cot rule and then follow up with the sine rule ;
(*b*) we can use the method given in (i) in reverse order ;
(*c*) we can use the tan $\tfrac{1}{2}(A - B)$ formula and then follow up with the sine rule ;
(*d*) we can use the ordinary cosine rule.

Of these possibilities I find that

(*b*) can lead to difficulties in sign,
(*a*) is preferable to (*c*) in the amount of working,
(*d*) is the quicker calculation if a table of square roots is available.

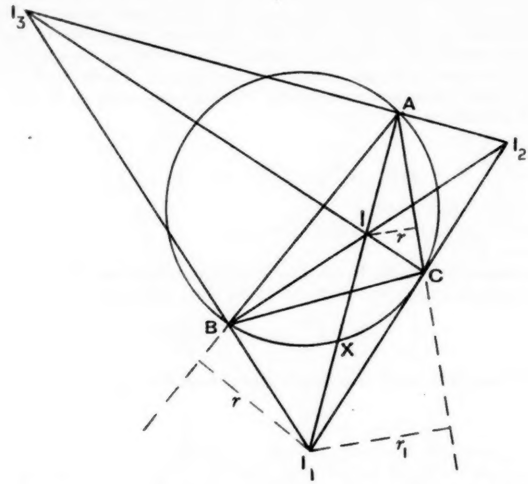
R. H. Peacock.

2107. *On Note* 1957.

The results obtained in this Note can be deduced from properties of the nine-points circle and the pedal triangle ; schoolboys find this exercise very interesting.

With the usual notation, I, I_1, I_2, I_3 form an orthocentric system, each point being the orthocentre of the triangle formed by the other three. Thus I is the orthocentre, and ABC is therefore the pedal triangle, of the triangle

$I_1I_2I_3$. The circle ABC is the nine-points circle of all four triangles formed by the points I, I_1, I_2, I_3. Thus the radius of the circumcircle of each of these four triangles is $2R$.



Let X be the mid-point of II_1; then X is on the nine-points circle of $I_1I_2I_3$; that is, X is on the circumcircle of ABC. Since X is the mid-point of II_1 and $\angle IBI_1 = \frac{1}{2}\pi$, X is the centre of the circle IBI_1C, and

$$XI = XI_1 = XB = XC = 2R \sin \tfrac{1}{2}A.$$
$$AI = r \operatorname{cosec} \tfrac{1}{2}A, \quad AI_1 = r_1 \operatorname{cosec} \tfrac{1}{2}A.$$
$$\angle I_2I_1I_3 = \tfrac{1}{2}\pi - \tfrac{1}{2}A, \text{ etc.}$$

The power of I for the circle $ABC = AI \cdot IX$.

Thus $$R^2 - OI^2 = r \operatorname{cosec} \tfrac{1}{2}A \cdot 2R \sin \tfrac{1}{2}A$$
$$= 2Rr,$$
or $$OI^2 = R^2 - 2Rr.$$
The power of $$I_1 = I_1A \cdot I_1X,$$
$$= r_1 \operatorname{cosec} \tfrac{1}{2}A \cdot 2R \sin \tfrac{1}{2}A,$$
whence $$OI_1^2 = R^2 + 2Rr_1.$$

Since I is the orthocentre of $I_1I_2I_3$ and $2R$ is the radius of its circumcircle,

$$II_1 = 2 \cdot 2R \cdot \cos I_1 = 4 R \sin \tfrac{1}{2}A;$$
$$I_2I_3 = 2 \cdot 2R \cdot \sin I_1 = 4R \cos \tfrac{1}{2}A, \text{ etc.}$$

Further, $$r = AI \sin \tfrac{1}{2}A = II_3 \cos I_2 \sin \tfrac{1}{2}A$$
$$= 2 \cdot 2R \cdot \cos I_3 \cdot \cos I_2 \cdot \sin \tfrac{1}{2}A$$
$$= 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C;$$
$$r_2 = CI_1 \cos \tfrac{1}{2}C = I_1I_3 \cos I_1 \cos \tfrac{1}{2}C$$
$$= 2 \cdot 2R \cdot \sin I_2 \cdot \cos I_1 \cdot \cos \tfrac{1}{2}C$$
$$= 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

Some remarks on orthocentric systems may be added here. Since any one point of the system is the orthocentre of the other three, and the system can have only one nine-points circle, it follows that the circumcircles of any combination of three points of the system are all equal, radius $2R$.

Imagine the four circumcircles as four circular discs of equal size. By sliding the discs into their positions as circumcircles of the four triangles, the distances between their centres and the central position O are $2R\cos A$, $2R\cos B$, $2R\cos C$. These are the distances of the vertices A, B, C from the orthocentre.

Each circle is the reflection of the circle ABC in its corresponding base. With a fixed base BC and a fixed angle A, the locus of the orthocentre H of the triangle ABC is the circle BHC. J. T. Davies.

2108. *The reciprocal nomogram.**

1. The simple nomogram for finding the effective resistance of two resistors in parallel is well known. Three axes are set up at angular intervals of 60°

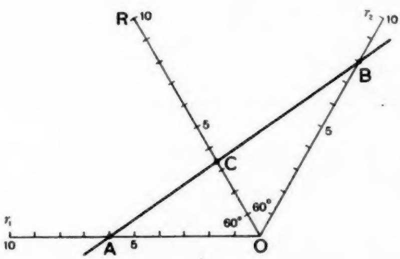


Fig. 1.

and furnished with identical linear scales (Fig. 1). A straight-edge determines three values which are related by the law

$$1/R = 1/r_1 + 1/r_2.$$

Proof. In the figure we have

$$\Delta AOB = \Delta AOC + \Delta BOC,$$

that is, $$\tfrac{1}{2}r_1r_2 \sin 60 = \tfrac{1}{2}r_1R\sin 60 + \tfrac{1}{2}r_2R\sin 60,$$

where r_1, r_2 and R represent the distances OA, OB and OC respectively. Dividing by $\tfrac{1}{2}r_1r_2R\sin 60$ we get

$$1/R = 1/r_1 + 1/r_2.$$

The percentage accuracy falls off for small values as we approach O.

For most purposes it will suffice if the axes are graduated from 0 to 10; for values outside this range, scale readings may be mentally multiplied by ten or by one hundred.

* The *Gazette* is much indebted to the Editor of *School Science Review*, and to Messrs. Kitchen and Moroney, for this shortened version of an article which appeared in *School Science Review*, Vol. XXIX, No. 107, October 1947. To this article readers are referred for further details, particularly on the optical applications.

The use of a diagram of this type for the reciprocal of a sum of reciprocals was described in *Gazette* X, p. 377, by Mr. D. F. Ferguson.

2. *Extensions.*

We have extended this basic nomogram in two ways:

(*a*) to the iterative problem of the type

$$1/R = 1/r_1 + 1/r_2 + 1/r_3 + \dots ;$$

(*b*) to the "Real is Positive" convention in geometrical optics.

2·1. *The iterative problem.*

The nomogram is now given *four* arms as in Fig. 2. Its use is best shown by illustration.

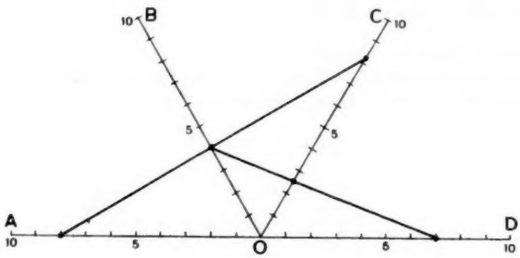


FIG. 2.

Example. Find R given by

$$\frac{1}{R} = \frac{1}{8} + \frac{1}{8\cdot 2} + \frac{1}{7}.$$

First, connect 8 on OA with 8·2 on OC to read off on OB the parallel resistance, 4. Then connect 4 on OB with 7 on OD to read off the final result, $R = 2\cdot 6$, on OC.

Evidently the same procedure may be applied to any number of components.

The chart may also be used in practice to select pairs of resistances to give a desired combined resistance. For example, what must be put in parallel with 70 ohms to give 26 ohms? By reference to Fig. 2 (mentally multiplying all scale readings by 10) the answer is found to be 40 ohms. Extensions to inductances and capacities will naturally suggest themselves to the reader.

It has been found useful in practice to colour the axes alternately red and black. Connecting two values on red axes then gives their combined resistance on the enclosed black axis, and similarly for two values on black axes.

The nomogram will also deal with the relation

$$\frac{a}{z} = \frac{b}{x} + \frac{c}{y}$$

by writing this in the form

$$\frac{1}{z/a} = \frac{1}{x/b} + \frac{1}{y/c}.$$

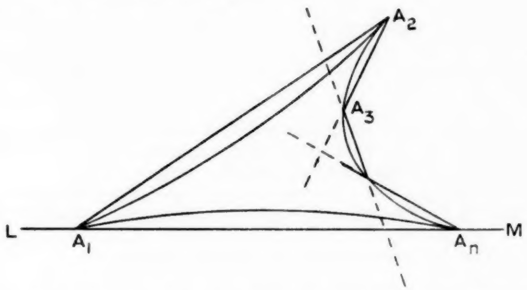
2·2. *The* "real is positive" *convention.*

Here we extend the nomogram so that the scales shown in Fig. 1 have now positive and negative parts. In this case we recommend that all positive axes be drawn in black and all negative axes in red. The axes are now labelled as U, V, F, according to the usual optical notation, and, following the "Real is Positive" convention, values on the black (positive) axes corre-

spond to real quantities and values on the red (negative) axes to virtual quantities. Solution or invention of problems proceeds exactly as before, and the reader will see that the nomogram can be used for a variety of lens problems. R. KITCHEN and M. J. MORONEY.

2109. *Curves of constant diameter.*

There have been several notes in the *Gazette* about the existence of curves of constant diameter other than the circle. The following is a way of generating such curves, of a very general nature.



Let $A_1A_2 \ldots A_n$ be a polygon, which is re-entrant at all except three vertices, say A_1, A_2, A_n, and has sides of lengths a_1, a_2, ... , a_n (starting with A_1A_2). Place a rod LM of length l along A_1A_n in such a position that A_1A_n lies within LM and

$$A_1L = \tfrac{1}{2}(l - a_1 + a_2 + a_3 + \ldots - a_n).$$

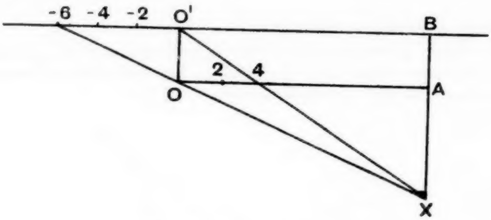
Rotate the rod about A_1 till it lies along A_1A_2, then about A_2 till it lies along A_2A_3, and so on. Then it can easily be shown that M finally comes to the initial position of L, and so the locus of the ends of the rod is a closed curve consisting of arcs of $2n$ circles, having a continuous gradient and a constant diameter of length l. The polygon has to be re-entrant in order that the rod shall have to rotate only through 180° before coming back to its original position. Clearly the figure can also be used by first using a compass to describe an arc centre A, radius AL, from L to N on AB produced, then to draw an arc centre B, radius BN, from BA to BC, etc., but the rod method is put first because of its more general application.

The sides of the polygon can be replaced by any arcs which are (i) convex inwards, (ii) such that in the process of rolling the rod round the new figure $A_1A_2 \ldots A_n$ in the way described below the rod turns counter-clockwise (or clockwise) continually. The rod must in this case be placed first along the tangent at A_1 to the arc A_1A_n, rotated about A_1 to the tangent to the arc A_1A_2 at A_1, then rolled along the arc A_1A_2 to the tangent to this arc at A_2, and so on. So long as AL has the initial value given above, where now a_1, a_2, ... a_n are the lengths of the arcs A_1A_2, ... , $A_{n-1}A_n$, it will be found that M comes finally to the initial position of L, and so the locus of L and M is again a curve of constant diameter consisting now of the arcs of $2n$ circles and arcs of $2n$ involutes of the curves forming the curvilinear polygon.

In the expression for A_1L the sign of a_r is opposite to that of a_{r-1} if A_r is not a re-entrant angle, but it is the same as that of a_{r-1} if A_r is a re-entrant angle, the sign of a_1 being minus when A_1 is taken to be one of the non-re-entrant angles. H. V. LOWRY.

2110. *Addition to Note* **1558.** "*Rate of Work*" *problems.*

The Note 1558 dealt with a graphical method for "rate of work" problems. Here is a slight extension.



If a tap will fill a bath (plug perfect) in 4 minutes, and if the plug in fact leaks so that (tap off) the bath empties in 6 minutes, how long will it take to fill the bath when the tap is on and the plug leaks?

The time required is OA. Incidentally, if OO' represents the capacity of the bath, then XA represents the water which leaked out and XB the total inflow.

The method is, of course, simply the transformation to

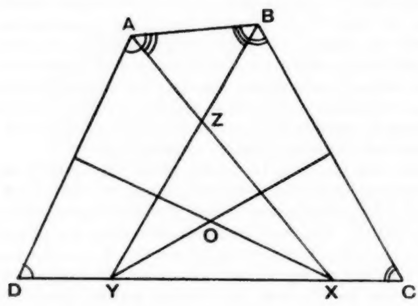
$$\frac{1}{4}+\frac{1}{(-6)}=\frac{1}{x}.$$

It suffers from the same drawbacks as the second method in Note 1558.

C. Dudley Langford.

2111. *A geometrical converse.*

The following proof was shown to me by a Dutch officer while I was in a Prisoner of War camp in Java. It does not appear to be well known in this country.



Given : a quadrilateral $ABCD$ with a pair of opposite angles supplementary.

To prove : that $ABCD$ is cyclic.

Construction : Draw AX, BY so that $\angle XAD = \angle D$, $\angle YBC = \angle C$, to meet DC in X, Y respectively. Let AX, BY meet in Z.

Proof. Since
$$\angle A + \angle C = \angle B + \angle D,$$
$$\angle A - \angle D = \angle B - \angle C.$$

Thus by construction $\angle ZAB = \angle ZBA$.

Hence the three triangles XAD, YBC, ZAB are all isosceles. It follows that the perpendicular bisectors of AD, BC pass through X and Y respectively. Let them meet in O. Since the triangles XAD, YBC are isosceles, XO and YO must be the bisectors of the angles X and Y, in the triangle XYZ.

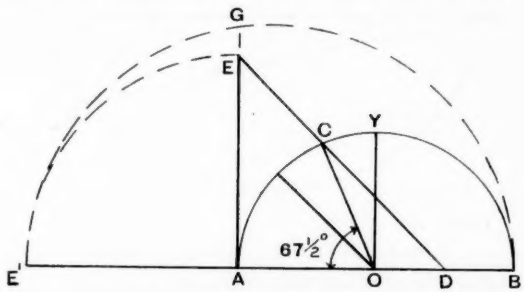
Hence OZ must be the bisector of the angle Z, and therefore OZ produced is the perpendicular bisector of AB, since the triangle ZAB is isosceles.

Hence a circle centre O can be drawn to pass through A, B, C, D.

N. J. F. Craig.

2112. *The circle squared!*

In Note 1826 (May 1945) Mr. B. A. Swinden gave a construction, the substance of which I quote below, for the approximate rectification of the circle. In the diagram the full lines represent the construction then given.



The right angle AOY is bisected and by a further bisection the angle AOC is obtained as $67\frac{1}{2}°$. Then OB is bisected in D, DC is joined and produced to meet the perpendicular AE in E.

When the radius $OA = r$, the Note showed that

$$\begin{aligned} AE &= 3r \sin 67\tfrac{1}{2}°/(1 + 2 \cos 67\tfrac{1}{2}°) \\ &= 1{\cdot}57007r \\ &= \tfrac{1}{2}\pi r \text{ (very nearly)}. \end{aligned}$$

There the Note ends. But suppose we continue as shown by the dotted lines, where $AE' = AE$, and on $E'B$ a semicircle is drawn and AE is produced to meet it in G.

Then
$$\begin{aligned} AG^2 &= AE' \,.\, AB \\ &= \tfrac{1}{2}\pi r \,.\, 2r \\ &= \pi r^2. \end{aligned}$$

Thus the square on AG is approximately equal to the area of the circle, with an error of only ·092 per cent.

Alfred J. J. Thomas.

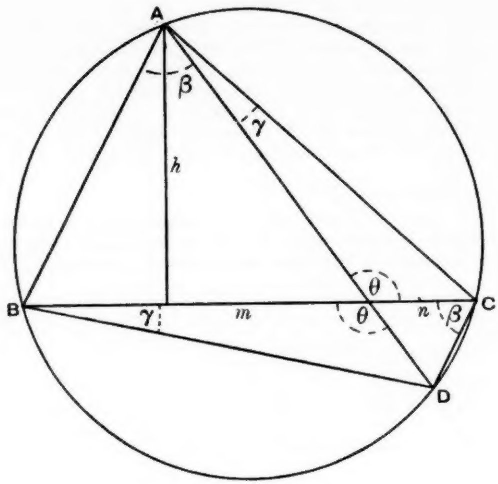
2113. *On Note* 2002.

The two identities

$$(m+n) \cot \theta = n \,.\, \cot B - m \,.\, \cot C, \qquad \text{(i)}$$

$$(m+n) \cot \theta = m \,.\, \cot \beta - n \,.\, \cot \gamma, \qquad \text{(ii)}$$

can be proved more shortly and neatly as follows.



Let $h = 1$. Then

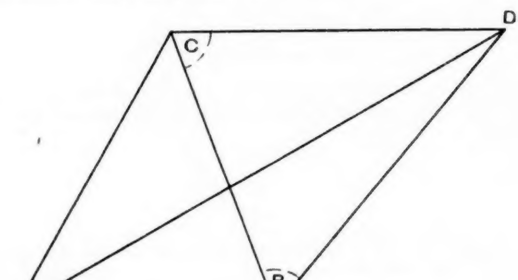
$$m = \cot B + \cot \theta, \quad n = \cot C - \cot \theta.$$

Multiplying by n, m respectively and subtracting, (i) follows, and (i) applied to the triangle BCD gives (ii), as is at once obvious from the figure.

A. Steiner.

2114. *The Brocard angle.*

Without using any formula for the Brocard points or the Brocard angle ω, we can prove that ω can not exceed $30°$.



Assuming $A \leqslant B \leqslant C$, $B + C$ is a minimum if $A = B = C$. The figure, in which $\angle DBC = \omega$ (see *Gazette*, July 1947, p. 133), shows that ω is a maximum if $\angle BCD = B + C$ is a minimum. Hence the maximum value of ω is $30°$.

A. Steiner.

2115. *The definition of pole and polar.*

The kind of geometry to which the following remarks apply is the kind in which there are plenty of lines which do not cut a given conic, plenty of points from which it is not possible to draw tangents to it, and in which a circle is a simpler curve than an ellipse.

In this kind of geometry there are certain difficulties about defining the polar of a point with respect to a given conic. There are three properties each of which is sometimes given as the definition of the polar p of a point P with respect to a conic S.

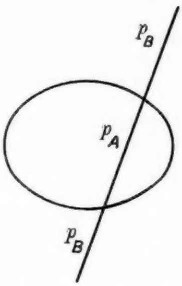


FIG. 1.

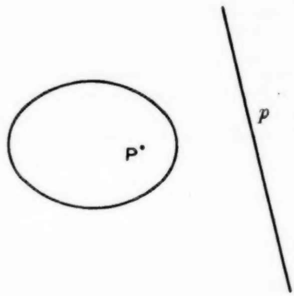


FIG. 2.

Definition (i). p is the chord of contact of tangents from P to S.

Definition (ii). p is the locus of the intersection of tangents to S at the ends of chords through P.

Definition (iii). p is the locus of points Q such that Q is the harmonic conjugate of P with respect to the points where PQ cuts S.

Taking the two figures 1 and 2 :

(i) applies to Fig. 1 but not to Fig. 2 ;

(ii) applies to Fig. 2 and to the part of p in Fig. 1 marked p_B ;

(iii) applies to Fig. 2 and to the part of p in Fig. 1 marked p_A.

It surely follows that none of the three definitions is wholly satisfactory.

In the case when S is a circle, centre O, there is a completely satisfactory definition of the polar of P, supposing that the inverse point to P, called Q, has first been defined.

Definition (iv). p is the line through Q at right angles to OQ, that is, to OP.

This only applies to a circle. It can be extended to any conic by saying that if p_0 is the polar of P_0 with respect to a circle S_0 and p_0, P_0, S_0 are projected into p, P and a conic S, then p is the polar of P with respect to S.

There is, however, a fifth possible definition which covers all cases, and which is very convenient as one from which to prove the other properties of p and P. It is perhaps a little difficult to word satisfactorily.

Definition (v). If P is the point (x_1, y_1) and p is the line given by the equation which is that of the tangent at P to S should P lie on S ; then whether P lies on S or not, p is the polar of P.

This is given much more succinctly for particular cases :

e.g. for $x^2+y^2=a^2$, the polar of (x_1, y_1) is $xx_1+yy_1=a^2$;

for $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the polar of (x, y) is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$,

and similarly for all conics.

This definition has the advantage that the reciprocal theorem—that if the polar of P passes through R, then the polar of R passes through P—follows at once, for if R is (x_2, y_2), taking the case of the ellipse the same equation

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$$

is the condition that p passes through R and that r passes through P.

From the reciprocal theorem and the fact that for a point on S the polar is the tangent, the properties used as definitions (i) and (ii) follow at once.

Thus, leaving the reader to draw the figure, if the tangents at L, M meet at P, then l goes through P, and so p goes through L, and similarly through M. Thus p is the line LM, the chord of contact. Again, if PRT is a chord through P and the tangents at the points R, T where it meets S meet at V, the polar of V, just proved to be RT, goes through P, and so the polar of P goes through V, and hence the locus of V is p.

To show now that (iii) agrees with (iv), let a chord through P be given by $x=x_1+r\cos\theta$, $y=y_1+r\sin\theta$, then this chord meets $x^2/a^2+y^2/b^2=1$ where

$$(x_1+r\cos\theta)^2/a^2+(y_1+r\sin\theta)^2/b^2=1\ ;$$

that is, $$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0,$$

so that, if r_1, r_2 are the roots,

$$\frac{1}{r_1}+\frac{1}{r_2}=-2\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)\Big/\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right).$$

Again, where the line meets the polar $xx_1/a^2+yy_1/b^2=1$ we have

$$\frac{(x_1+r\cos\theta)x_1}{a^2}+\frac{(y_1+r\sin\theta)y_1}{b^2}=1,$$

whence $$r\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0,$$

so that $$\frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2},$$

which is the condition that the chord should be divided harmonically.

Definition (iv) for the circle also follows at once, since the gradient of OP is y_1/x_1, while that of $xx_1+yy_1=a^2$ is $-x_1/y_1$, and the perpendicular from O to $xx_1+yy_1=a^2$ is of length

$$a^2/\sqrt{(x_1^2+y_1^2)}=a^2/OP.$$

It seems therefore a pity that (v) is seldom mentioned as a possible definition of the polar.

C. O. TUCKEY.

2116. *Variations on a very simple theme.*

So long ago that no one but myself is the least likely to remember it,* the *Mathematical Gazette* printed a note from me in which I explained how the figure of the common trapezium could be used as a peg on which to hang questions in geometry suitable as tests for pupils of various ages from twelve to twenty.

* Vol. XII, No. 170 (May 1924), p. 108, Note 706.—Ed.

Now that all up-to-date teachers have learned to think of, or at least to talk of, elementary mathematics as being one subject, it may be worth while to show how the same figure (or indeed almost any other) can be used as a link in the chain to bind the various branches of the subject together.

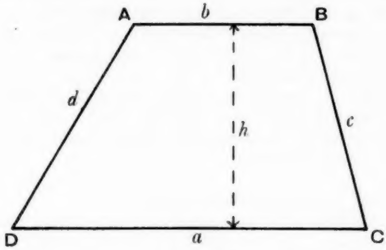


FIG. 1.

ABCD, Fig. 1, is a trapezium with a and b as the parallel sides at a distance h apart.

It is a little doubtful nowadays whether one is allowed to ask a pupil to *prove* any result which is likely to be useful, but I suppose that even the most alternative syllabus will permit the pupil to be set, behind the closed doors of the classroom, to " prove the formula $A = \frac{1}{2}h(a+b)$ for the area of the trapezium ". This proof is geometry, but when proved we can turn to that favourite first question on an algebra paper " calculate b given the other letters " or " change the subject of the formula to b ".

" Construct a square equal to the trapezium " is geometry, but " if the side of the square is the geometric mean of a and b prove that h is their harmonic mean " is algebra, while " calculate the side of the square, for numerical values of a, b, h, to 4 decimal places " is arithmetic.

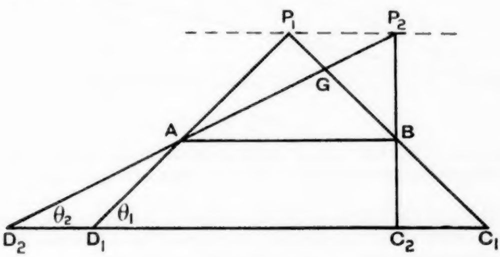


FIG. 2.

To bring in trigonometry, take two special cases (Fig. 2) in which ABC_1D_1 has $BC_1 = AD_1$ and ABC_2D_2 has $\angle B = \angle C_2 =$ a right angle. " For given values of a, b, h calculate θ_1 and/or θ_2 " is trigonometry ; " Prove that

$$\tan(\theta_1 - \theta_2) = h(a-b)/\{(a-b)^2 + 2h^2\}\text{ "}$$

brings in the addition theorems ; while " in Fig. 1 express $\cot C + \cot D$ in terms of a, b, h " brings in the cotangent.

D

To go back to geometry, we have from Fig. 2 the problems "AB being fixed, h constant and CD of given length, prove that the locus of intersection of CB and DA is a straight line" or again, "in Fig. 2 prove that $AG = 2GP_2$".

For somewhat more difficult questions we may set "Given a, b, c, d construct the trapezium" (geometry), depending on the triangle whose sides are d, c, $a - b$, or its sequel "Given a, b, c, d calculate the area" (algebra-cum-trigonometry). One more geometry question is: "Given that $2h = a + b - c - d$ prove that two circles can be inscribed to touch each other, one to touch BA, AD, DC and the other BA, BC, CD".

Finally for a touch of mechanics: "If in Fig. 2 D_1ABC_1 is the velocity-time graph for a train between two stations, find what fraction the average speed is of the full speed." It should be noted that this question cannot be repeated for D_2ABC_2 without risk. C. O. T.

2117. *Variations on an examination question.*

Probably many readers of the *Gazette* will have seen the last question of the Geometry paper set January 1948 for London Matriculation and may have discussed it with their pupils; it would, however, be interesting to know how many of them considered at all carefully the implications of the figure involved and the different versions possible for similar questions on the same figure.

The question was as follows, with the letters changed round, more for convenience than any idea of dodging copyright restrictions:

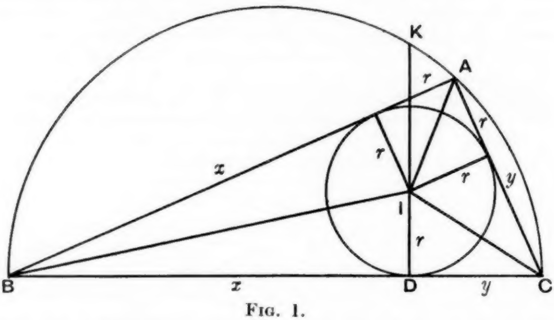


FIG. 1.

"The triangle ABC has $\angle A$ a right angle and the radius of its inscribed circle is r. Express the perimeter and the area of the triangle in terms of r and a."

A glance at Fig. 1, in which $x + y = a$, will show that the perimeter is $2(r + a)$, and the usual method for proving $\Delta = rs$ shows that the area is $r^2 + ar$. This answers the question as set; now for some variations.

Perhaps the neatest result which emerges from Fig. 1 is that $\Delta = xy$. This result can be got by saying that the area is both $\frac{1}{2}AB \cdot AC = \frac{1}{2}(r + x)(r + y)$ and also is $r(r + x + y)$. Or, again, it comes from using $BC^2 = BA^2 + AC^2$, which gives

$$(x + y)^2 = (r + x)^2 + (r + y)^2,$$

whence

$$2xy = 2(r^2 + rx + ry).$$

There ought, of course, to be also some neat geometrical construction to prove the result. Failing this method, which eludes me, another idea is to

use the question as an argument in favour of combining trigonometry with geometry, and to say that the area is always $xy \cot \frac{1}{2}A$ for $xy = (s-b)(s-c)$ and $\cot \frac{1}{2}A = \sqrt{\{s(s-a)/(s-b)(s-c)\}}$, so that $xy \cot \frac{1}{2}A$ reduces to Hero's formula for Δ, and when $\frac{1}{2}A = 45°$, $xy \cot \frac{1}{2}A$ is xy.

Again, since the area is xy, if, as in Fig. 1, DK is drawn at right angles to BC to meet the circle BAC at K, it follows that $\Delta = DK^2$, so the question might be set thus :

" The in-circle of a right-angled triangle touches the hypotenuse at D, and from D a perpendicular DK is erected to meet the circumcircle at K. Prove that DK^2 is equal to the area of the triangle."

Another allied question is : " In Fig. 1, given x and y, calculate r." This merely asks for the solution of the quadratic $r^2 + r(x+y) = xy$, but it raises for the teacher the question whether he can make up any neat solution to fit this equation with integers.

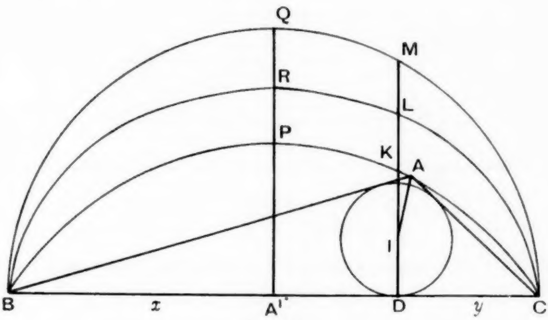


FIG. 2.

Since $\Delta = DK^2$ in Fig. 1 when $\cot \frac{1}{2}A = 1$, it is tempting to consider Fig. 2 when $\cot \frac{1}{2}A < 1$ and ask whether the value $xy \cot \frac{1}{2}A$ again leads to DK^2.

This is not a question which the ordinary matriculation candidate could answer except by accurate drawing and measurement, but it might be handed over to the specialists.

In Fig. 2, for the isosceles triangle with vertical angle A, that is, for the triangle BPC, the area is $BA' \cdot A'P$, and therefore is $A'R^2$ where $A'R$ is the mean proportional between $A'P$ and $A'Q$. Now for the triangle ABC, $xy = DM^2$ and the multiplier $\cot \frac{1}{2}A$ is the same as for the triangle BPC, so if the ellipse whose semi-axes are $A'C$ and $A'R$ is drawn and DK meets it at L, then since $DL^2 : DM^2 = A'R^2 : A'Q^2$ it follows that the area of the triangle ABC is not DK^2 but DL^2. C. O. T.

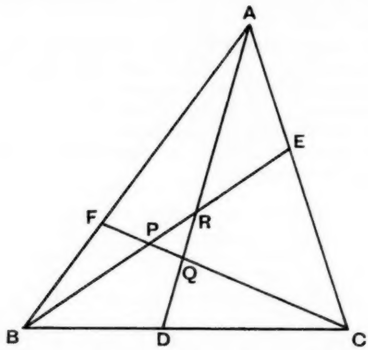
2118. *On Menelaus' Theorem, Ceva's Theorem and the harmonic property of a quadrilateral.*

The intimate relation among the ancient theorems of collinearity and concurrency due to Menelaus and Ceva and the comparatively modern theorem on the harmonic property of a quadrilateral is elucidated in the present note with the help of another theorem which is believed to be new. This theorem which connects the three theorems aforesaid is enunciated below :

Theorem. ABC is any triangle ; D, E, F are any three points on the sides of BC, CA, AB respectively ; and AD, BE, CF form the triangle PQR where

P is the intersection of BE and CF, Q that of CF and AD, and R that of AD and BE. Then

$$(AD, QR) = (BE, RP) = (CF, PQ) = \frac{BD \,.\, CE \,.\, AF}{DC \,.\, EA \,.\, FB}.$$



Proof. Regarding BE as a transversal to the $\triangle ADC$, we have by Menelaus' theorem :

$$\frac{AR \,.\, DB \,.\, CE}{RD \,.\, BC \,.\, EA} = -1. \qquad \text{.................................(1)}$$

Regarding CF as a transversal to the $\triangle ABD$, we have again

$$\frac{AQ \,.\, DC \,.\, BF}{QD \,.\, CB \,.\, FA} = -1. \qquad \text{.................................(2)}$$

From (1) and (2) we get

$$(AD, QR) = (BD \,.\, CE \,.\, AF)/(DC \,.\, EA \,.\, FB).$$

Similarly (BE, RP) and (CF, PQ) are also equal to

$$(BD \,.\, CE \,.\, AF)/(DC \,.\, EA \,.\, FB).$$

Ceva's theorem follows immediately from the above result when AD, BE, CF are concurrent and P, Q, R coincide and the cross-ratios reduce to unity. The converse also follows with equal ease.

When D, E, F are collinear

$$BD \,.\, CE \,.\, AF = -DC \,.\, EA \,.\, FB,$$

and therefore each of the cross-ratios (AD, QR), (BE, RP), $(CF, PQ) = -1$, *i.e.* each of the lines AD, BE, CF is divided harmonically by the other two. This is the harmonic property of the quadrilateral formed by the sides of the $\triangle ABC$ and the transversal DEF. A. K. Srinivasan.

2119. *The line $x + y = 0$ as the line at infinity.*

The most commonly used special systems of homogeneous coordinates are those in which the lines $x + y + z = 0$ and $z = 0$ are interpreted as the line at infinity. The system in which the line $x + y = 0$ is interpreted as the line at infinity is trivial by comparison, but offers some reward to the seeker after light mathematical entertainment.

1. *Euclidean interpretation.* The vertex $C(0, 0, 1)$ of the triangle of reference ABC lies on the line at infinity. Let P be the point (x_1, y_1, z_1) and I the unit point. Let CP meet AI at L, BI at M, and AB at N. Then since AP is $y_1 z = z_1 y$ and AI is $z = y$,

$$A(P, I, B, C) = (z_1/y_1, 1, 0, \infty) = z_1/y_1.$$

But
$$A(P, I, B, C) = (P, L, N, C) = NP/NL.$$

Thus $z_1/y_1 = NP/NL$, and similarly $z_1/x_1 = NP/NM$. We then have

$$x_1 : y_1 : z_1 = NM : NL : NP.$$

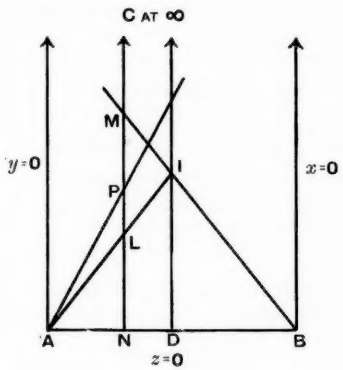


Fig.. 1.

The point at infinity on AB is $(1, -1, 0)$. Consequently D, the midpoint of AB, is $(1, 1, 0)$ and I lies on CD. Let us make $ID = DA$, and further let us make CA perpendicular to AB. Then

$$x_1 : y_1 : z_1 = NB : NA : NP,$$
$$\tan PAB = z_1/y_1, \quad \tan PBA = z_1/x_1.$$

2. *Gradient of* $lx + my + nz = 0$. The last two relations given above afford a method of dealing with angles. The line AX through A parallel to

$$lx + my + nz = 0$$

is $lx + my + nz + \lambda(x + y) = 0$, where $l + \lambda = 0$, that is, $y(m - l) + nz = 0$. Therefore $\tan XAB = (l - m)/n$, which is therefore the tangent of the angle between $lx + my + nz = 0$ and AB.

3. *The circular points.* If P is any point on the circle on AB as diameter, $\tan PAB$ is the reciprocal of $\tan PBA$, and therefore the equation of this circle is $xy = z^2$. The circular points are obtained by solving this equation and the equation $x + y = 0$ simultaneously. The coordinates obtained are $(\pm i, \mp i, 1)$.

The corresponding line equation is

$$l^2 + m^2 + n^2 - 2lm = 0.$$

4. *Illustrations.* The use of the foregoing is illustrated by two simple examples.

Example (a) *If Q_1 and Q_2 are the points of contact of the two members of a*

coaxal system of circles which touch a given line, then Q_1Q_2 subtends a right angle at a limiting point of the system.

Each circle of the system of coaxal circles having A and B as limiting points passes through the remaining four points of intersection of the circular lines through A and B. The equation of the system is therefore, for variable λ,

$$x^2+z^2+\lambda(y^2+z^2)=0.$$

The corresponding line equation is

$$\lambda(1+\lambda)l^2+(1+\lambda)m^2+\lambda n^2=0.$$

Let λ_1, λ_2 be the values of λ for the two circles touching (l_1, m_1, n_1). Then they are the roots of the quadratic

$$\lambda^2 l_1^2+\lambda(l_1^2+m_1^2+n_1^2)+m_1^2=0.$$

Q_1 is the point $\{\lambda_1(1+\lambda_1)l_1,\ (1+\lambda_1)m_1,\ \lambda_1 n_1\}$.

Therefore $$\tan Q_1AB \tan Q_2AB=\frac{\lambda_1 n_1}{(1+\lambda_1)m_1}\cdot\frac{\lambda_2 n_1}{(1+\lambda_2)m_1},$$

and this, on substitution for $\lambda_1\lambda_2$, $\lambda_1+\lambda_2$, reduces to -1. Hence the angle subtended by Q_1Q_2 at A is a right angle.

Example (*b*). *A pair of common tangents of two members of a coaxal system of circles and their chords of contact touch a conic whose foci are the limiting points of the coaxal system.*

Each conic of the system of confocal conics having A and B as foci touches the circular lines through A and B. The line equation of the system is therefore, for variable μ,

$$l^2+m^2+n^2+\mu lm=0.$$

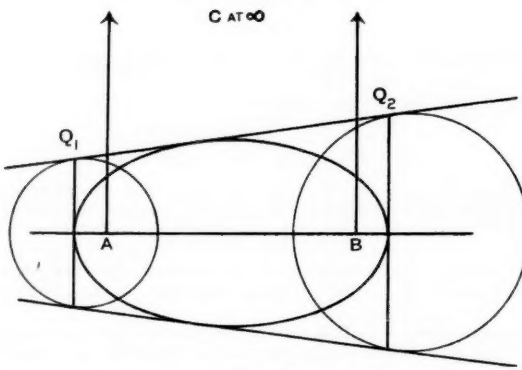


FIG. 2.

The conic of the system touching the lines $(l_1, m_1, \pm n_1)$ is

$$l_1m_1(l^2+m^2+n^2)-(l_1^2+m_1^2+n_1^2)lm=0. \quad \text{.....................(i)}$$

It has been found in Example (*a*) that if the lines $(l_1, m_1, \pm n_1)$ are common tangents of two circles of the coaxal system having A and B as limiting points, the points of contact are

$$\{\lambda(1+\lambda)l_1,\ (1+\lambda)m_1,\ \pm\lambda n_1\},$$

where λ is given by the quadratic

$$\lambda^2 l_1^2 + \lambda(l_1^2 + m_1^2 + n_1^2) + m_1^2 = 0. \quad \text{.......................(ii)}$$

The chords of contact of the common tangents to these two circles are therefore $m_1 x - \lambda l_1 y = 0$, where λ is given by (ii). The condition that these should touch (i) is

$$l_1 m_1 (m_1^2 + \lambda^2 l_1^2) + \lambda m_1 l_1 (l_1^2 + m_1^2 + n_1^2) = 0,$$

that is,
$$l_1 m_1 \{\lambda^2 l_1^2 + \lambda(l_1^2 + m_1^2 + n_1^2) + m_1^2\} = 0.$$

We see from relation (ii) that this is satisfied. Hence the conic (i) touches the common tangents and the chords of contact. WINSTON SNOWDON.

2120. *Every spherical triangle is isosceles.*

Let ABC be a spherical triangle whose sides are unequal. We prove that $b = c$.

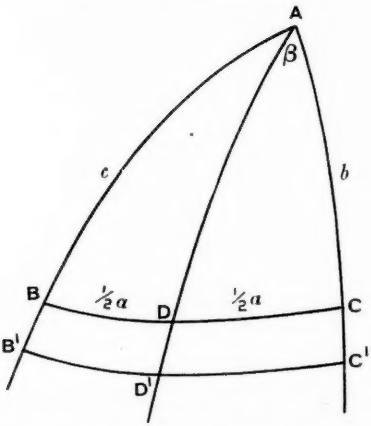
If D is the middle point of BC, then, by the sine rule,

$$\sin BD/\sin BAD = \sin AB/\sin ADB$$

and
$$\sin CD/\sin CAD = \sin AC/\sin ADC,$$

so that, on dividing and using the notation of the figure,

$$\sin \beta/\sin \alpha = \sin c/\sin b. \quad \text{...............................(i)}$$



Now give the base BDC a small variation to the position $B'D'C'$ by letting the points B, C, D slide, as it were, down AB, AD, AC. The angles α, β are unchanged in this displacement. Let us adjust the variation, as we can, so that

$$\delta c \equiv BB' = k \sin c, \quad \delta b \equiv CC' = k \sin b,$$

where k is very small.

By logarithmic differentiation of equation (i), whose left-hand side is constant in this variation, we have

$$\cos c / \delta c / \sin c = \cos b / \delta b / \sin b.$$

Hence, on inserting the values of δc and δb,

$$\cos c = \cos b,$$

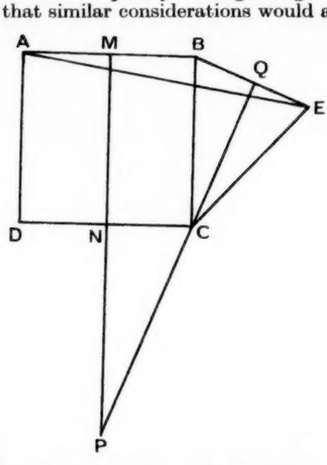
so that $$c = b.$$

E. A. MAXWELL.

2121. *That every angle is a right angle.*

Glancing through the *Gazette* (XXXI, No. 296; see p. 221) I was struck by the phrase " Angles are as real as men ", in the article on " Mathematics and the Child " by C. Gattegno. Hoping that the misprint that never arrives had come at last I read further, and came to the passage which is the purpose of this note: " In the well-known problem which leads to the conclusion that a right angle is equal to an acute angle, when we want to prove that there is a mistake, there is no other way of proving the falsity of the result than by drawing a figure using a circle and ruler. It is not mere reasoning which is used, but action (the construction)."

Without wishing to get involved in a philosophical argument which I could not sustain, I give a *proof* (in what I take to be the ordinary accepted sense of the word) that the result referred to is in fact erroneous. I add that the phrase " well-known problem " does not specify it completely, and the form which I know best leads to the equality of a right angle with an *obtuse* angle; but I have no doubt that similar considerations would apply in other cases.



In the usual description of the figure, which we have taken the liberty of drawing correctly, $ABCD$ is a square, and the line CE is drawn outward, equal in length to a side of the square. The perpendicular bisectors of AB, AE meet in P, and by skilful adjustment the angle ECP, *measured from CE through CB and CD to CP*, is made to appear less than two right angles. On the basis of this figure, the angles ECD, ADC are proved equal.

Consider, however, an alternative way of drawing the figure. Let $ABCD$ be a square, and the line CE drawn outward, as before, equal in length to the side of the square. Let the bisector of the angle BCE meet BE in Q and the perpendicular bisector of AB in P. Clearly PQ is the perpendicular bisector of BE, so that P is the centre of the circle ABE. Hence P is on the perpendicular bisector of AE, being thus identified with the point P described in the preceding paragraph.

But the angle ECP, measured from CE through CB and CD to CP, is *reflex*, since it is equal to the " straight angle " QCP *plus* $\frac{1}{2}\angle BCE$, the fact of addition being in the very nature of euclidean geometry as ordinarily understood.

I suppose it might be argued that the last step is ultimately an appeal to experiment ; and in that sense perhaps the whole of euclidean geometry is experimental. But I doubt whether that was what the writer meant in the passage which I have quoted. I certainly have not used a compass and ruler.

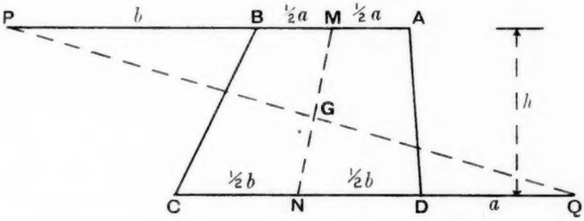
It may be worth while to add that the " usual proof " that every triangle is isosceles breaks down for similar reasons, but the flaw is more obvious. The internal bisector of the angle A and the perpendicular bisector of the side BC of a triangle ABC are " drawn " to meet inside the triangle. But they really meet in the middle point of the arc BC of the circumcircle.

E. A. MAXWELL.

2122. *The centre of gravity of a trapezium.*

1. It appears that an elementary proof of the well-known drawing-office construction for the centre of gravity of a trapezium is not readily available. The method below has the additional advantage of stressing the utility of replacing a triangle by equal masses at its vertices—a result most undergraduates seem to have forgotten.

2. *Construction.* In the figure, M, N are the midpoints of AB, CD and $BP = CD$, $DQ = AB$; then G is given by the intersection of MN and PQ.



Proof. G obviously lies on MN. Replace the triangles ABD, BCD by equal masses $ha/6$ at A, B, D, $hb/6$ at B, C, D. Then the trapezium is equivalent to masses $\frac{1}{3}h(a+\frac{1}{2}b)$, $\frac{1}{3}h(b+\frac{1}{2}a)$ at points K, L in AB, CD where KGL is a straight line. Thus

$$GM : GN = GK : GL = (b+\tfrac{1}{2}a) : (a+\tfrac{1}{2}b) = PM : QN.$$

3. The following elegant alternative solution, due to Dr. J. A. Todd, is unfortunately beyond the scope of most of the scientific students who use the construction.

If G_1, G_2 be the centres of gravity of the triangles BCD, ABD, then G lies on G_1G_2 and MN. But the two straight lines PG_1D and BG_2Q form a conic containing the hexagon PBG_1G_2DQ with Pascal line MN.

4. Incidentally, the method of § 2 has another effective technical application in hydrostatic stability problems, particularly when the displacements from equilibrium of the floating bodies are not necessarily small.

For example, the standard values $h\tan\theta$ and $\frac{1}{2}h\tan^2\theta$ for the displacements of the centre of buoyancy of a rectangular solid tilted through a finite angle θ can be written down immediately. The normal textbook derivation is much longer.

M. APPLEBY.

2123. *Two corrections.*

1. Mr. D. F. Ferguson has recently brought to light mistakes in Shank's celebrated (1873) calculation of π to 707 decimal places. I have just come across an astonishing error in another usually unimpeachable source; and since notes on integral-sided geometrical figures have frequently appeared in the *Mathematical Gazette* in past years, I think some of your readers may be interested in this one.

On p. 210 of Vol II of L. E. Dickson's *History of the Theory of Numbers* appears the statement that in 1814 J. Cunliffe, searching for a triangle with rational area and rational angle-bisectors, found such a triangle (39, 150, 175) by placing in juxtaposition two each of the three integral-sided right-angled triangles 7, 24, 25; 24, 32, 40; and 24, 143, 145. A figure is given.

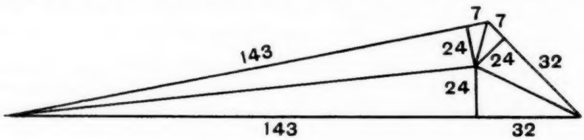


FIG. 1.

But it is a delusion and a snare. The right-angled triangles quoted, when juxtaposed, do not form a triangle at all, but an irregular hexagon with one re-entrant angle.

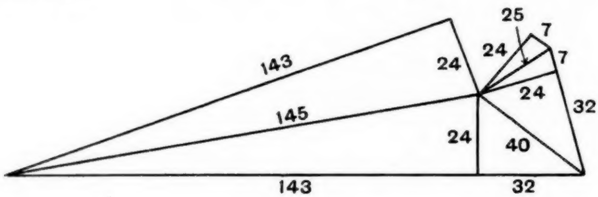


FIG. 2.

Even the true triangle 39, 150, 175 has irrational area, and only one rational angle-bisector.

2. In Vol. II, p. 505, a set of four integers such that the sum of the squares of any three is itself a square is given as 186120, 102120, 32571, 23838; the last number should be 23828. ALFRED G. STRIPP.

2124. *The new ball in cricket.*

The long discussions which have gone on in the Test Match commentaries concerning the value of the new ball and the great advantage that it gives to a side with an accurate fast attack prompted me to investigate the mathematical basis of the problem.

In order to simplify the problem I have considered two types of bowling only.

1. In new ball bowling I assume that the ball is delivered without spin with the seam in a vertical plane at a small angle to the direction of motion. In theoretical flow this would cause a small movement of the rear stagnation point, since the air would tend to break away at the discontinuity in the surface caused by the seam, thus causing a deflection of the air relative to the ball as shown in Figure 1.

Considering the horizontal motion to be an arc of a circle, the force at right angles to the direction of motion will be kv^2, where v is the velocity of projection and k is a constant whose value is determined by the amount of

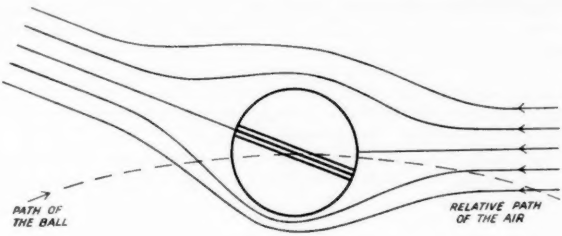


Fig. 1.

movement of the rear stagnation point which has been induced by the seam. The horizontal equation of motion is therefore

$$kv^2g = Wv^2/r,$$

and so the radius of curvature of the path is given by

$$r = W/kg,$$

a result which is independent of the velocity of projection. The amount of swerve can be measured by the distance x through which the ball has been deviated from its straight path in travelling 66 ft. From Figure 2 this is given by

$$x(2r - x) = 66^2.$$

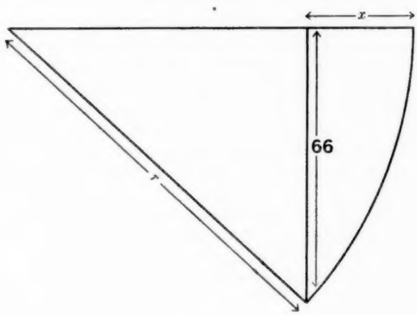


Fig. 2.

Since x is a very small quantity compared with r, x^2 is negligible, and so we have

$$x = 66^2/2r = 66^2kg/2W.$$

Owing to viscosity, which has been neglected in the foregoing, the value of k will be reduced, and so the amount of swerve will be reduced. This is because the path of the air is longer on one side of the ball than the other. As the shine is worn off the new ball, this reduction in swerve progressively increases as the viscosity has more effect.

The conclusions of this elementary treatment of the problem are therefore (*a*) the amount of the swerve is the same whether the bowler is bowling fast or slow; (*b*) the faster the bowler the shorter the time that the batsman has to react to the swerve, and this constitutes the greater danger of the fast attack; (*c*) because of the shine on the ball, spin will have the effect of reducing the swerve because it displaces the seam from the vertical plane without producing a circulation on its own account; (*d*) the amount of the swerve can be materially increased by the method of delivery, *e.g.* jerking the hand, a method not allowed in cricket, can give an added circulation round the ball, and this will give a greater force at right angles to the direction of motion as illustrated by "pitching" in baseball.

2. When the ball has had the shine worn off it, then a circulation can be set up by spinning the ball. The force at right angles to the direction of motion is now

$$kv\Gamma,$$

where Γ is the amount of circulation set up by the spin. Hence the horizontal equation of motion is

$$kv\Gamma g = Wv^2/r,$$

from which we find that the radius of curvature of the path is given by

$$r = Wv/k\Gamma g.$$

As before, the amount of swerve is given by

$$x = 66^2/2r = 66^2 k\Gamma g/2Wv.$$

When bowling with an old ball the conclusions are (*a*) the amount of swerve varies inversely as the speed of delivery, *i.e.* a slow bowler can make a ball swerve more than a fast one; (*b*) roughness and spin will increase Γ and so increase the amount of swerve. R. H. PEACOCK.

2125. *Scale drawing and the LBW rule in cricket.*

In showing the relation between two quantities by means of a graph their two scales can be chosen independently, and advantage is often taken of this arbitrariness. With due regard to units, properties of area or gradient hold whatever the scales may be. Geometrically, the relative scales along the two axes may be altered by orthogonal projection, and the invariant nature of ratios of lengths along parallel lines and of areas is well known. It may not be commonly realised that the same procedure may be applied to scale-drawing.

An example relating to cricket, which should interest boys, was suggested to me by Dr. R. A. Lyttleton. For the purpose of investigating geometrically the implications of the LBW law, some device of the kind to be described is almost indispensable for the following reasons. The rectangular area "between wicket and wicket" is the zone of chief importance in the LBW rule. Since this rectangle is 22 yards long and only 9 inches wide, a scale-drawing of it might be $5\frac{1}{2}$ inches long and hence *on the same scale* only $\frac{1}{16}$ inch wide. The drawing of lines almost parallel to the two longer sides, to represent the plan of paths of the ball, would produce a diagram from which it would be quite impossible to read off results with any worth-while precision.

The difficulty is overcome by *increasing the scale for width.* A satisfactory diagram, for example, is got by taking the *width scale* as $\frac{8}{9}$ of the actual size; this gives a rectangle $5\frac{1}{2}$ inches long and 2 inches wide for the area "between wicket and wicket". The projected path of any ball can be traced on such a diagram, and straight balls (that is, those which do not deviate laterally) will be represented by straight lines. The effects, for LBW purposes, of a

bowler delivering the ball from different distances from the wicket at his end (the permitted range is about 4 feet on either side of the wicket) are made clear by such a diagram. The proportion of deliveries of any particular type which, apart from height, satisfy the conditions for LBW may be obtained from such a diagram without using calculations which might be too hard for the average cricketer or schoolboy.

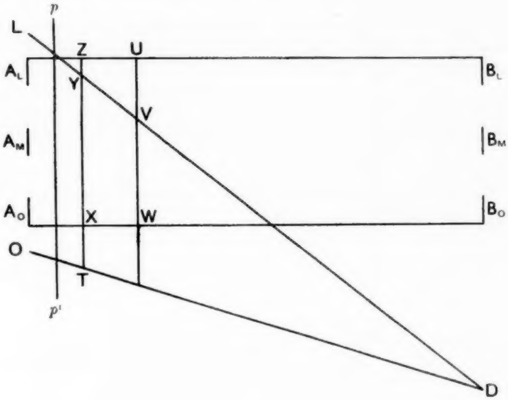


Fig. 1.

Fig. 1 is the diagram for straight right-hand bowling from 9 in. wide of the bowler's wicket ; Fig. 2 is for straight left-hand bowling from 2 ft. 3 in. wide of the bowler's wicket. In each diagram $A_L A_M A_O$ denote the three stumps at the batsman's end, $B_L B_M B_O$ those at the bowler's end ; pp' is the popping crease, 4 ft. in front of the wicket ; D is the point directly under the point of release of the ball. DL, DO are the lines along which the *centre of the ball* must travel if the leg or off stump is just to be grazed (diameter of ball 2·8 in.).

Suppose a batsman plays well forward so that his leg is 8 ft. from his wicket when struck by a ball pitching 16 ft. from the wicket. Assuming a uniform distribution of deliveries across the wicket, the following figures are obtained under the present LBW rule, under which the ball (1) must pitch between wicket and wicket on the off side, (2) must be going to hit the wicket, (3) must hit a part of the batsman between wicket and wicket.

Condition satisfied by balls.		*Proportion that satisfy the LBW conditions.* *Fig.* 1.	*Fig.* 2.
(i) Balls pitching " between wicket and wicket "	VW/UW	·63	·15
(ii) Balls that would have hit the wicket	XY/YT	·78	·57
(iii) Balls that hit a part of the batsman between wicket and wicket	XY/XZ	·89	·65

The bowler, as a rule, moves away from the line of the wickets, and by the time the ball reaches the batsman (about 1 second after delivery) is some feet away from this line and therefore well off either of the diagrams. This feature must make it very difficult for the bowler to form an accurate judg-

ment whether the batsman is out or not. If the ball is actually between wicket and wicket when it hits the batsman, it will seem to the bowler to be in line with a point outside the leg stump. The same is true for any other fielder except the wicket-keeper, and he must almost always be unsighted by the batsman when LBW incidents arise.

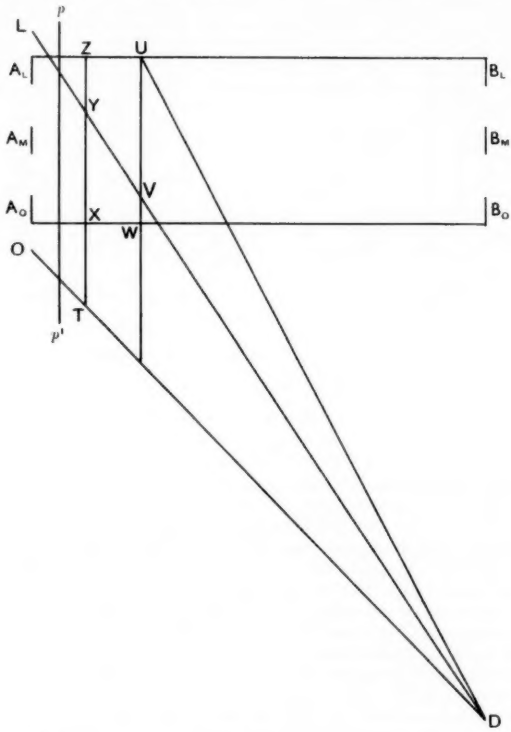


Fig. 2.

It is possible in a similar way to make approximate calculations for other types of bowling and to consider what would be the effect of any proposed changes in the rules.

E. D. Tagg.

2126. *Illustrations in the use of crosses.*

The theorems below have been shown in various ways by Lawlor (*Gazette*, October 1920) or by Gibbins (*Gazette*, October 1939), or both. Also Roger Johnson, *Modern Geometry*, may be consulted.

1. *A, B, C, P are four coplanar points, no three collinear ; D, E, F are images of P in the sides of the triangle. Then the circles BDC, CEA, AFB cut on the circle DEF.*

For, let PD, PE, PF cut the sides BC, CA, AB respectively in D_1, E_1, F_1,

and let the circles BDC, CEA cut in R. We show that the circle DEF goes through R.

For, using crosses, $\measuredangle DFE = DFP + PFE$,

$$DFP = D_1F_1P = D_1BP = DBD_1 = DBC = DRC.$$

Similarly, $PFE = CAE = CRE$.

Hence $$DFE = DRC + CRE = DRE.$$

2. Now bisect all intervals from P. The circle BDC becomes the nine-points circle of PBC; the circle DEF becomes the pedal circle of P for ABC. Hence the nine-points circles of the triangles PBC, PCA, PAB and the pedal circle of P for ABC meet in a point. Thus:

The four nine-points circles of the triangles PBC, PCA, PAB, ABC meet in a point (for the possibility that they meet in threes in four distinct points is easily shown not to occur), and hence, by symmetry:

The pedal circles of each of the four points with respect to the triangle formed by the other three also go through the point where the nine-points circles meet.

3. Some account of the above has already been given in my *Higher Course Geometry*, p. 232 ff. We now go further with the same methods.

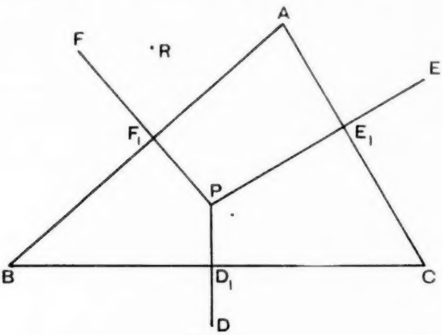


Fig. 1.

If in Fig. 1 Q is the point isogonal to P, and S is isogonal to R, then we prove that Q, S are inverse points for the circle ABC.

We first note that any point X is given when two of the three crosses BXC, CXA, AXB are known. Now if X, X' be inverse points for a circle, centre O, and likewise Y, Y' and Z, Z', then, using crosses (since X, X', Y, Y' are concyclic),

$$OXY = X'Y'O, \quad OXZ = X'Z'O,$$

$$YXZ = YXO + OXZ = OY'X' + X'Z'O = Z'X'Y' + Y'OZ'.$$

Hence $$YXZ + Y'X'Z' = Y'OZ' = YOZ.$$

In particular, if Y, Z, W are on the circle of inversion, then

$$YXZ + YX'Z = YOZ = 2YWZ.$$

Again, if P, Q be isogonal points for ABC, then by definition,

$$ABP = QBC, \quad PAB = CAQ.$$

Hence $$APB = ABP + PAB = QBC + CAQ = BQA + ACB.$$

Thus $$APB + AQB = ACB.$$

Now, in Fig. 1, if P, Q are isogonal, and R, S isogonal, we have

$$APB+AQB=ACB, \quad ASB+ARB=ACB.$$

But $$ARB+APB=AFB+APB=0,$$

and so $AQB+ASB=2ACB$, and similarly.

Whence Q, S are inverse points for the circle ABC. It is easily shown that Q is the centre of the circle DEF.

4. We may note that the pedal triangles of inverse points (for the circumcircle of the given triangle) are inversely similar.

For if X, Y, Z be the feet of perpendiculars from P to BC, CA, AB and X_1, Y_1, Z_1 the feet of perpendiculars from the inverse point P', then

$$YXZ=YXP+PXZ=YCP+PBZ=ACP+PBA=CAB+BPC,$$

$$Y_1X_1Z_1=CAB+BP'C.$$

Now $$BPC+BP'C=2BAC.$$

Hence $$YXZ+Y_1X_1Z_1=0.$$

5. The feet of the perpendiculars from B to CP, PA, AC will be denoted by B_1, B_3, B_4; the feet of the perpendiculars from C to BP, PA, AB will be denoted by C_1, C_2, C_4.

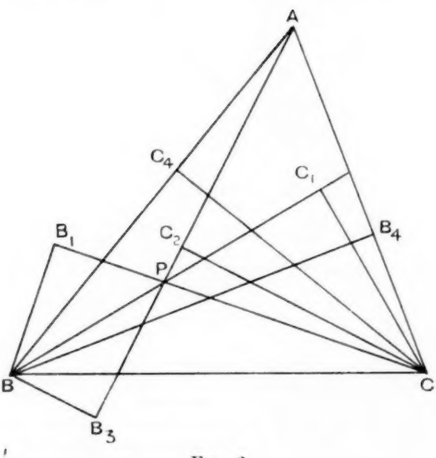


FIG. 2.

Then B_1C_4, C_1B_4 meet on AP.

For $B_1C_4BC_1B_4C$ is a hexagon in a circle, and hence by Pascal's theorem, B_1C_4 and C_1B_4 meet on a line joining the cut A of C_4B and B_4C to the cut P of BC_1 and CB_1; that is, they meet on PA, at L, say. Thus $B_1B_3B_4$ and $C_4C_2C_1$ are in perspective from L.

6. We saw in § 2 that the pedal circle of B for CPA and of C for APB cut in T, say. We shall show that *the other cut is* L, the point in § 5.

For $LB_1B_4=C_4B_1B_4=C_4BB_4=ABB_4=AB_3B_4=LB_3B_4$.

Hence L, B_1, B_3, B_4 are concyclic; similarly L, C_1, C_2, C_4 are concyclic.

7. *The pedal triangles are directly similar.* For

$$B_1B_3B_4=B_1B_3B+BB_3B_4=B_1PB+BAB_4=CPB+BAC,$$

$$C_4C_2C_1=C_4C_2C+CC_2C_1=C_4AC+CPC_1=BAC+CPB.$$

The centre of similitude is the point I where the pedal circles and the nine-points circles all cut.

This theorem is connected with that on the perspectivity of the triangles, if we note that if AOB and A_1OB_1 are straight lines through one cut O of two circles, the other cut being O_1, then $AO_1B = A_1O_1B_1$.

8. *Pascal's theorem.* We have used this theorem for the case of the circle. It is, of course, usually regarded as a theorem of projective geometry for a conic. The fact that a large portion of euclidean geometry can be based on the notion of the " cross " makes the proof of the theorem for a circle, by means of crosses, something more than an unnecessary exercise.

The proof is given in my *Higher Course Geometry*, p. 13, where a figure will be found.

Let $AB'CA'BC'$ be the hexagon, and let BC', $B'C$ cut in D, and CA', $C'A$ cut in E. The circles $B'C'D$, $C'A'E$ cut on DE by the pivot theorem, at P, say. Let $B'A$, $A'B$ cut DE in F, F_1.

Then
$$A'PF = A'PE = A'C'E = A'C'A = A'B'A = A'B'F,$$
$$B'PF_1 = B'PD = B'C'D = B'C'B = B'A'B = B'A'F_1.$$

Hence the circle $A'B'P$ cuts DE in F and F_1. Hence $F = F_1$.

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2127. *Lemoyne's theorem.*

The pedal circles with respect to a triangle of points on a straight line are such that there is a certain point whose power is the same for all the circles. (They are orthogonal to some fixed circle, real or " imaginary ".)

An ingenious proof of this is given by Gallatly, *Modern Geometry of the Triangle*, pp. 58, 59, but it is spoilt by divagations and the introduction of a parabola. The following is merely a revision.

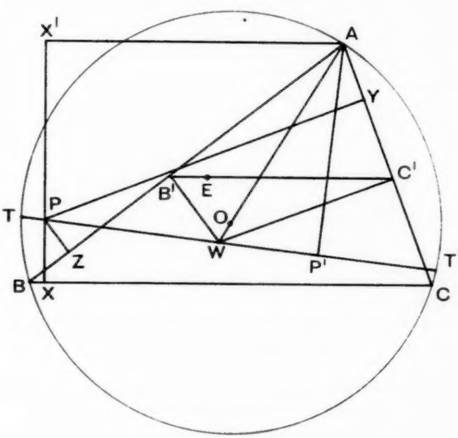

Fig. 1.

Let the given line cut the circle ABC in T, T'. Let P be any point on the line, X, Y, Z the feet of its perpendiculars on BC, CA, AB. Let the join of A be the centre O of the circle ABC cut TT' in W and let B', C' be the feet

of the perpendiculars from W to AB, AC. Let YZ, $B'C'$ meet in E. The circle $AB'C'$ on diameter AW, and the circle AYZ on diameter AP, both go through P', the foot of the perpendicular from A to TT'. By the four-circles theorem,* P', Y, E, C' are concyclic. Hence using crosses, $P'YC' = P'EC'$.

If X' is the foot of the perpendicular from A to PX, then A, Y, P', X' lie on a circle, diameter AP; hence $P'YC' = P'X'A$. Hence $P'X'A = P'EC'$.

But $B'C'$ is easily seen to be parallel to BC and hence to AX'. Hence P', E, X' are collinear.

Since P' is fixed, therefore E, X' describe similar ranges, as P moves; hence so do E, X and hence XE goes through a fixed point S on the perpendicular from P' to BC, with $XE/ES = X'E/EP'$.

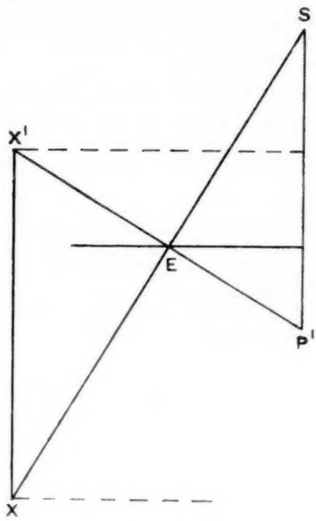

Fig. 2.

Let XS cut the pedal circle XYZ again in S', then

$$XE \, . \, ES' = YE \, . \, EZ = X'E \, . \, EP',$$

since A, Y, P', Z, X' lie on a circle with diameter AP. Hence X', P', X, S' are concyclic points.

Let P_1' be the image of P' in the right bisector of XX', then

$$XS \, . \, SS' = SP' \, . \, SP_1'.$$

This is the power of S for the circle XYZ. Hence this power is independent of the position of P. Hence Lemoyne's theorem.

Corollary 1. If TT' cuts the sides BC, CA, AB in L, M, N, then taking P at these points, the powers of S for the circles on AL, BM, CN as diameters are all equal. Hence S is on the radical axis of these circles, which is, of

* If we wish to avoid this, we may put instead

$$P'YE = P'YZ = P'AB' = P'C'B' = P'C'E.$$

course, the line of orthocentres of the four triangles formed by BC, CA, AB and LM.

2. If we take P at T or T' then X, Y, Z lie on the Simson line of the point, and thus the proof shows that these Simson lines meet in S. Recall the construction for S. We draw AP' perpendicular to TT' and through the foot P' a perpendicular to BC. Then S is on that perpendicular. Thus, by symmetry, if we draw perpendiculars from A, B, C to TT' and from their feet draw perpendiculars respectively to BC, CA, AB, these meet in S; that is, S is the *orthopole* of TT' for ABC.

3. If TT' goes through O, then $B'C'$ bisects XX'. Hence P', S are images in $B'C'$, and S coincides with P_1'. Hence all the pedal circles go through S. In particular S is on the pedal circle of O, the nine-points circle of ABC.

Note. From the point of view of the axiomatic treatment of crosses, Lemoyne's theorem is " deeper " than most of those in Note 2126. We used the rectangle property of the circle, which is on the same level as Desargues' theorem on perspective triangles with parallel sides. This comes from Pappus' theorem in the case when the opposite sides of the hexagon are parallel. The proof given of this in my recent note, October 1947, involves the drawing of an auxiliary line, while the proof, not given there, of Desargues' theorem involves several auxiliary lines.

I believe that *that* is why the facts lie deeper. H. G. Forder.

2128. *Three short proofs.*

I have nowhere seen the following proofs, though they seem obvious enough.

1. ***The Simson line of P for the triangle ABC bisects the join of P to the orthocentre H of ABC.***

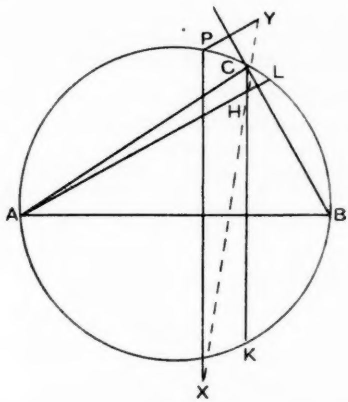

Fig. 1.

Let X, Y be the images of P in AB and BC, and let AH and CH cut the circle in L, K. Then as H, K are images in AB, we have, using crosses,

$$\sphericalangle XHK = HKP = CKP = CBP, \quad YHL = ABP,$$
$$XHY = XHK + KHL + LHY = CBP + ABC + PBA = 0.$$

Thence X, H, Y are collinear, and as the Simson line bisects PX, PY, it bisects PH.

This also follows from Pascal's theorem, which shows that H is on the join of the cuts of PK, AB and of PL, BC.

2. *An ancient Chinese theorem.*

Let A, B, C, D be concyclic and I_1, I_2, I_3, I_4 be the incentres of the triangles BCD, CDA, DAB, ABC. Then it is well known that the bisectors of the

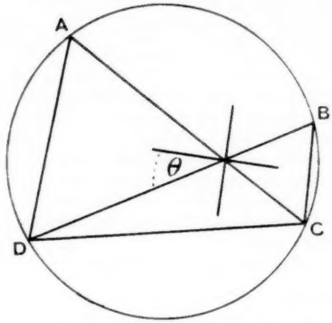

FIG. 2.

angles at A, B meet in the midpoint X of the arc CD, and that $XI_1 = XI_2$. Hence I_1I_2 strikes these bisectors at equal angles, and hence it strikes AC, BD at equal angles, θ, say. Consideration of the figure then shows that:

I_1, I_2, I_3, I_4 are the vertices of a rectangle whose sides I_1I_2, I_2I_3, I_3I_4, I_4I_1 are parallel to the bisectors of the angles between BD and AC.

If r_1, r_2, r_3, r_4 are the corresponding in-radii, then

$$r_2 + r_4 = I_2I_4 \sin\theta, \quad r_1 + r_3 = I_1I_3 \sin\theta,$$

whence $r_1 + r_3 = r_2 + r_4$, an ancient Chinese theorem.

3. The sides of a triangle ABC cut a line in X, Y, Z, and at these points perpendiculars to the sides are drawn: these form a triangle $A'B'C'$. Then *the join of the orthocentres of ABC, $A'B'C'$ is bisected by XYZ.*

FIG. 3.

This was shown by N. M. Gibbins geometrically in *Gazette*, 1925, p. 440, and analytically in *Gazette*, 1944, p. 107. For another proof see Goormaghtigh, *Gazette*, 1946, p. 293, where a list is given of papers in *Mathesis*, inaccessible

to me. Of all the proofs, including the one below, I prefer the first listed, as the most illuminating.

But all we need show is that the distances of the orthocentre of the triangles from the line are equal. Now the orthocentre is the mean centre of weights $\tan A$, $\tan B$, $\tan C$ at the vertices ; hence its distance from the line is

$$(p_1 \tan A + p_2 \tan B + p_3 \tan C)/(\tan A + \tan B + \tan C). \quad \text{.........(i)}$$

Using dashes to denote things which relate to the second triangle, the distance of *its* orthocentre from the line is

$$(p_1' \tan A' + p_2' \tan B' + p_3' \tan C')/(\tan A' + \tan B' + \tan C'). \quad \text{.......(ii)}$$

Now $A' = \pi - A$; we shall show that the numerators in (i) and (ii) are equal apart from sign. Let BC, CA, AB cut the line in X, Y, Z, at angles also denoted by X, Y, Z.

Then $\qquad YZ/\sin A = AZ/\sin Y = p_1/\sin Y \sin Z,$

$$p_1 \tan A = p_1 \sin A/\cos A = YZ \sin Y \sin Z/\cos A.$$

For the other triangle

$$Y' = \tfrac{1}{2}\pi - Y, \quad Z' = \tfrac{1}{2}\pi - Z.$$

$$p_1' \tan A' = YZ \,.\, \sin Y' \sin Z'/\cos A' = -YZ \cos Y \cos Z/\cos A.$$

$$p_1 \tan A + p_1' \tan A' = -YZ \cos(Z - Y)/\cos A = -YZ.$$

We now take account of the sign of $\tan C$ and use $XZ = XY + YZ$.

H. G. Forder.

2129. *Algebraical identities.*

I. The following collection may be of use in constructing examples for the young. They are from various sources ; in particular L. E. Dickson's *History of the Theory of Numbers* has been drawn upon. We write them in *non*-homogeneous form when this does not spoil them, and this and other simplifications of the formulae in Dickson sometimes disguises their purport. Thus, for example, if we write a^3 for a in 13 and make the identity homogeneous, we get a recipe for finding three cubes whose sum is a square. Again, 20 gives a recipe for finding three fourth powers whose sum is a square.

I am indebted to Mr. B. I. Hayman for checking these formulae.

1. $a^2 + (a+1)^2 + a^2(a+1)^2 = (a^2 + a + 1)^2.$
2. $(a^2 - 1)^2 + a^2(a+2)^2 + (2a+1)^2 = 2(a^2 + a + 1)^2.$
3. $(a^2 - 2)^2 + (a^2 + 4a + 2)^2 = 2(a^2 + 2a + 2)^2.$
4. $(a^2 + 2a - 1)^2 + (a^2 - 2a - 1)^2 = 2(a^2 + 1)^2.$
5. $(a^2 + 6a + 7)^2 + (a^2 + 2a - 1)^2 = 2(a^2 + 4a + 5)^2.$
6. $(7a^2 - 12a + 7)^2 = 4(3a^2 - 7a + 3)^2 + 13(a^2 - 1)^2 =$ sum of three squares.
7. $a^2 + b^2 + c^2 + (a + b + c)^2 = (b + c)^2 + (c + a)^2 + (a + b)^2.$
8. $3\{(a + 2b + 1)^2 + (b + 2a + 1)^2 + (a - b)^2\}$ $= (3a + 1)^2 + (3b + 1)^2 + (3a + 3b + 2)^2.$
9. $(a^2 + b^2 + c^2 + d^2)^2 = (a^2 - b^2 - c^2 + d^2)^2 + 4(ab - cd)^2 + 4(ac + bd)^2.$
10. $4\{ab(x^2 + y^2)\}^2 + 4\{xy(a^2 - b^2)\}^2 + \{(x^2 - y^2)(a^2 - b^2)\}^2$ $= \{(x^2 + y^2)(a^2 + b^2)\}^2.$
11. $(x^2 + y^2 + z^2)\{(ax + by)^2 + (a^2 + b^2)z^2\}$ $= (ax^2 + bxy + az^2)^2 + (by^2 + axy + bz^2)^2 + (ay - bx)^2z^2.$
12. $4(a^2 + a + 1)^3 - 27a^2(a+1)^2 = (a - 1)^2(2a + 1)^2(a + 2)^2.$
13. $(a + 1)^3 + (2 - a)^3 + 27a = 9(a + 1)^2.$
14. $a(a + 2)^3 + (2a + 1)^3 + 27a^2 = (a^2 + 7a + 1)^2.$

15. $a(a-3)^2+(3a-1)^2=(a+1)^3$.

16. $2\{(a+b+c)^3-27abc\}=(a-b)^2(a+b+7c)+(b-c)^2(b+c+7a)$
$\qquad+(c-a)^2(c+a+7b)$.

17. $(a^3-1)^3+(1+3a-a^3)^3+27a^3(1+a)^3=9a(1+a+a^2)^3$.

18. $2(a^2+a+1)^2=a^4+(a+1)^4+1$.

19. $2(a^2+a+1)^4=(a^2-1)^4+(2a+1)^4+a^4(a+2)^4$.

20. $16a^2(a-1)^4+16a^2(a+1)^4+(a^2-1)^4=(a^4+14a^2+1)^2$.

21. $16a^2+4a(a-1)^2+(a^2-1)^2=(a+1)^4$.

22. $(ka+1)^5+(ka-1)^5+8(ka)^5=k^2a(k^2a^2+1)^2$, when $k=10$.

23. $(2ka+1)^6-(2ka-1)^6-(ka+2)^6+(ka-2)^6=k^6a^5-k^2a$, when $k=360$.

24. $(a+b+c)^7-(a+b-c)^7-(a-b+c)^7-(-a+b+c)^7$
$\qquad=56abc\{3(a^4+b^4+c^4)+10(b^2c^2+c^2a^2+a^2b^2)\}$.

25. $(a+b+c-d)^4+(a+b-c+d)^4+(a-b+c+d)^4+(-a+b+c+d)^4$
$\qquad=4(a^2+b^2+c^2+d^2)^2$
$\qquad+16\{(ab-cd)^2+(bc-ad)^2+(ca-bd)^2\}$.

26. $\{a(d+c)-b(c-3d)\}^4+\{2(bc-ad)\}^4+\{a(d-c)-b(c+3d)\}^4$
$\qquad=\{a(d-c)+b(c+3d)\}^4+\{2(bc+ad)\}^4+\{a(d+c)+b(c-3d)\}^4$.

II. The following question from Smith's *Algebra* (p. 602, No. 26) often causes trouble.

If $x+y+z=0$ and $x^3/(b-c)+y^3/(c-a)+z^3/(a-b)=0$, then

$$(b-c)(b+c-2a)^2/x^2+(c-a)(c+a-2b)^2/y^2+(a-b)(a+b-2c)^2/z^2=0.$$

A direct attack leads to much work. The following is an identity; in the unwritten terms circulate x, y, z and λ, μ, ν.

$$\begin{aligned}\lambda(\mu-\nu)^2y^2z^2+\ldots+(\lambda x+\mu y+\nu z)(\mu\nu x^3+\ldots)\\ =(x+y+z)(\lambda x+\mu y+\nu z)(\mu\nu x^2+\ldots)-\lambda\mu\nu(x+y+z)(x^3+y^3+z^3)\\ -xyz\{\mu\nu(\mu+\nu)x+\ldots\}+\lambda\mu\nu(x^4+\ldots-2y^2z^2-\ldots\}.\end{aligned}$$

If $x+y+z=0$ and $\lambda+\mu+\nu=0$, all terms on the right-hand side vanish. Take $\lambda=b-c$, $\mu=c-a$, $\nu=a-b$, and the result follows at once.

H. G. Forder.

2130. *Iterative determination of a familiar particular integral.*

Let p_m be a polynomial in x of degree m, and let f be a polynomial, with constant coefficients, in the operator D, that is, d/dx, subject only to the condition that D is a factor of f. Also, for arbitrary n, let

$$y_n=p_m+fp_m+f^2p_m+\ldots+f^np_m.$$

Then identically

$$\text{(i)}\ y_{n+1}=y_n+f^{n+1}p_m,\qquad \text{(ii)}\ y_{n+1}=p_m+fy_n.$$

If $n\geqslant m$, then $f^{n+1}p_m=0$, since D is a factor of f; hence from (i), $y_{n+1}=y_n$, and therefore from (ii), y_n satisfies the equation $(1-f)y=p_m$. In other words,

$$y=p_m+fp_m+f^2p_m+\ldots+f^mp_m$$

is a particular integral of this linear differential equation.

This is, of course, the ordinary solution, but it is obtained without any manipulation of D as a variable in a symbolic algebra. E. H. N.

REVIEWS.

Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability. Edited by Jerzy Neyman. 42s. Pp. 501. 1949. (University of California Press: Cambridge University Press)

This book contains the papers which were delivered at two symposia held at Berkeley, California, " to mark the end of the war and to stimulate the return to theoretical research ", the first being held 13–18 August, 1945, and the second 27–29 January, 1946. A special grant of the Administration of the University of California enabled the symposia to be held, and special funds were made available to assist the publication of the *Proceedings*.

Professor Jerzy Neyman, who held an appointment in the University of London up to 1938 and whose name, coupled with that of Egon S. Pearson, is well known in connection with a theory of statistical inference which has gained wide currency among statisticians, has edited the volume in his capacity as Professor of Mathematical Statistics and Director of the Statistical Laboratory of the University of California (Berkeley Campus). There are twenty-nine papers, and because each one of these would merit in the ordinary way publication in a scientific journal, this is a book that no statistician can afford to be without. Outside the class of statisticians, mathematical and otherwise, the attention of mathematicians is directed to a very sound and stimulating paper on the " Philosophical Foundations of Probability ", by Hans Reichenbach, and to another by Harold Hotelling, an authority on the subject, on " The Place of Statistics in the University ". This latter paper, together with the accompanying discussion, should be read by all heads of mathematical departments (and by university administrators) who are facing the need to-day to set up departments of statistics, or at any rate to arrange that their mathematical departments are strengthened by the addition of lecturers qualified to teach statistics.

The remaining papers vary very much in their content and character, as is indeed to be expected from symposia where papers on the applications of statistics were invited as well as papers on the mathematical theory. As a whole, the papers serve to show that the study of statistics can, and should, be carried on simultaneously in the same place in two directions: (*a*) the further development of mathematical statistical tests and tools for use by the experimenter, and (*b*) the application of recognised techniques by the experimenter and the reporting by him of his experimental problems for the further stimulation of the mathematical statistician to renewed efforts to cope with difficulties. There is not space to detail titles and authors, but it is at any rate of interest to remark that papers on the application of statistical methods deal with the fields of Physics, Psychology, Economics, Strategy, Astronomy, Forestry, Agronomy, Meteorology, Silviculture, Animal Breeding and Entomology. With the possible exception of Strategy, the items on this list represent fields of university study, to which a few more could be added, where a knowledge by the workers of statistical methods does much to assist in the design and analysis of the scientific experiments on which they are engaged, and in the proper interpretation of their results. J. Wishart.

Analytic Geometry. By Robin Robinson. Pp. ix, 147. 13s. 6d. $2.25. (McGraw-Hill). *Answers to Problems*. Pp. 12. 6d. 1949. (McGraw-Hill)

This book, which is described by the author as " a brief text for a conventional course in analytic geometry ", gives some account not only of lines, circles and conics, but of planes and quadrics besides. It would hardly be possible in 140 far from crowded pages to give more than an outline of the chief properties of the curves and surfaces described, but within the limits

necessarily imposed by the size of the book the author achieves his chief object, which is to provide a clear but concise presentation of the principal methods of the subject.

Intended for use in American colleges, the treatment is rather different from what we are accustomed to; and English readers might find parts of the book rather unsatisfactory. For example, having obtained the equation of the tangent to a circle by writing down the line at right angles to the radius, the author cites the equation as an example of the rule " Replace x^2 by xx_1 ... ", after which the rule is used to provide tangent equations for the conics. A verification of the rule for the ellipse is obtained later by showing that the middle points of chords of gradient m lie on a diameter, after which the coordinates of the points of contact of the tangents with gradient m can be found, and the equations of the tangents written down. Calculus is not used in the book.

The two chapters on three-dimensional analytical geometry are very pleasing, excellent use being made of $u + \lambda u'$ methods. Incidentally, it is not until the equation of a line in space is required that parameters are even mentioned.

An adequate number of exercises for the student is provided, and answers are available separately. R. W.

Principles of Mathematical Physics. By W. V. Houston. Second edition. Pp. xii, 363. $5. 1948. (McGraw-Hill)

The first edition of this book was published in 1934, and now, after fourteen years, it is followed by a second edition, very similar to the first. The preface claims that it was written for juniors, seniors and first-year graduate students. Most British students in their first two honours years would find it useful, but a book which starts at the beginning of vector analysis on page 145 is too elementary for third-year honours work. And a post-graduate student in physics who is told on page 209 that there is a function $F = U - TS$ which " is often called the free energy but is perhaps more appropriately called the work function ", and who never meets the function again or is told anything of its significance except (p. 227) in an " analogous " form, would scarcely be called a complete physicist. The scope of the book embraces a discussion of differential equations up to and including the method of Frobenius for expansion in series. A thorough account of the mechanics of particles involving Hamilton's Principle, and the Euler Variational technique, is followed by a chapter on rigid bodies. Next there is a short and unconvincing chapter on Thermodynamics, followed by an even less satisfying account of Statistical Mechanics, in which nothing later than the time of Josiah Willard Gibbs finds any place. The last 130 pages include a good and careful exposition of electricity and the electromagnetic field. In this part of the work, the mksc (metre–kilogram–second-coulomb) units are employed. Here particularly are some excellent diagrams, and on the whole the story is well told. The familiar pitfalls regarding B and H are neatly circumnavigated, though it will come as a shock to some people to find magnetostatics and the electromagnetic field introduced by these words: " We shall take as the fundamental law of force between elements of current

$$\mathbf{dF} = \frac{\mu_0}{4\pi}\frac{II'}{R^3}\,\mathbf{dl} \times (\mathbf{dl}' \times \mathbf{R})."$$

It is a pity the expression is never simplified in the familiar pictorial way.

The book is beautifully got up, and there are very few errors. It would be wise, however, if a third edition is ever contemplated, to alter the sentence

on p. 154 : " If the vector is a linear function of the time, each component is proportional to the time."

But when all this is said, there is still something that is much more serious. Is it really fair to give a student a book with the words " Mathematical Physics " in its title which has no mention of quantum theory, wave mechanics, hydrodynamics, sound or the standard functions of mathematical physics? And in which only five pages in all are concerned with any form of optics or wave motion? Perhaps the book would then be too long, but in its present form it hardly does justice to present-day physics. To have to discuss Statistical Mechanics without the quantum theory is like *Hamlet* without the Prince of Denmark : and physics has changed a lot since 1895. The electron is not even referred to in the index of this book. If we had lived in 1905, when the equivalence of mass and energy with which this book concludes was just being worked out, we should have relished this account. It comes to us now almost like something from the past. C. A. Coulson.

Living Mathematics. By R. S. Underwood and F. W. Sparks. Second edition. Pp. x, 374. 24s. 1949. (McGraw-Hill)

The first edition of this unusual book, published in 1940, was fully reviewed and criticised in the *Mathematical Gazette*. It was described as a cross between a textbook and " Mathematics for the Million ". The book has been thoroughly revised and many of the criticisms met. Much of the flippancy, tiresome in excess, has been toned down, and many practical examples added. As a self-educator it has clearly been improved. Whether it would fit any course followed in this country is doubtful. But the freshness of presentation remains. Elementary Calculus is headed " Variables caught in Action " Mathematical Induction is still " A Ladder without a Top " ; and there is a curious little chapter at the end, not in the first edition, which deals with extensions to Cartesian Geometry by the addition of further axes (in the same plane), the case taken being the simplest one of three axes (x, y, z) at $60°$ to each other, with consequent invariant relations between the coordinates of a point, which can, of course, be expressed in an infinity of ways. It must be emphasised, however, that the book is elementary, introducing trigonometry, calculus, logarithms, fractional indices, etc., and rarely reaching out beyond a Certificate course. H. V. S.

Maths is Fun. By Joseph Degrazia. Pp. 159. 8s. 6d. 1949. (Allen & Unwin)

This is a collection of puzzles, old and new, easy and difficult. Each chapter consists of variations on a theme. Generally a classic is taken first, *e.g.* " Wolf, Goat and Cabbage " or " The Seven Sevens ", and is worked out completely, to be followed by other problems of the same type, some worked out (in the answer pages), to others only a clue being given. The method leads to a certain monotony in working through the book, but it renders the volume a useful reference book for teachers requiring puzzles of a certain type. A fourth former would find it a delightful prize, though many of the problems are really difficult. The chapter on " Faded Documents " (beginning with the " Seven Sevens ") is good. Rather disappointing is the following one on " Cryptograms ". The classic " Send + More = Money " is worked out, but instead of many others of the same type we are given " Send + More + Gold = Money ", with the information that thirty-two solutions are possible. Surely the whole point of such problems is that there should be a unique solution : and they are easy enough to invent. (For instance, the reviewer's telephone no. is Bec : " Ebor ", in which Robe − Ebor = Bec and the remainder is an exact cube.)

But this minor criticism must not obscure the fact that the book is excellent value, should be possessed, and that if your favourite problem is not there, there is sure to be one something like it. We await with interest the promised successor which will include geometrical puzzles. H. V. S.

Arithmetical Essays. By PEDRO A. PIZÁ. Pp. 173. $3. 1948. (Soltero, Santurce, Puerto Rico)

This is a work wherein the specialist in mathematics and even the amateur will find much to engage their attention. The author has by means of arithmetical tables dealt fully with the sums of numbers, sum of squares, sum of cubes and higher powers by these simple tables : while on pp. 226–8 he shows how to obtain, by what he considers a simpler process than that of Bernoulli, the sum of the 10th powers of the numbers from 1 to 1000. We are next introduced to Kummer coefficients, escalator numbers and Glennie triangles. All these will repay the reader who is interested in matters of this kind.

The book is very nicely printed, and the illustration at the beginning showing the first printing of Pascal's triangle is very interesting and attractive.

J. T.

Anwendung der elliptischen Funktionen in Physik und Technik. F. OBERHETTINGER und W. MAGNUS. Pp. vii, 126. DM. 15.60. 1949. Die Grundlehren der mathematischen Wissenschaften, 55. (Springer, Berlin)

The enormous literature on elliptic functions is perhaps weakest on the applications to other topics in pure and applied mathematics. Halphen, partly fragmentary, and Greenhill, defective in arrangement, are in any case out of date ; Fricke's promised third volume never appeared. There is room, therefore, for a collection of up-to-date applications, particularly in modern physical and technical problems, and this new volume in Springer's admirable *Grundlehren* series gathers together many results which hitherto could only be found by ransacking periodical literature.

There are five chapters. The first merely collects the formulae needed in the later sections, and calls for little comment, save perhaps to deprecate the use of zn u for Jacobi's Zeta-function

$$\int_0^u \mathrm{dn}^2 u \, du - u\, E/K \, ;$$

notation should distinguish between the doubly-periodic and the pseudo-periodic functions.

The second chapter deals with conformal representation, for rectangles and ellipses with their associated Green's functions, and for polygons. The third contains electrical applications, and the fourth brings in quite recent developments in hydrodynamics and aerodynamics. If we have, for example, a rectangular boundary at zero potential, enclosing charges, a suitably disposed double series of images formed by reflections in the sides of the rectangle will provide a solution of the boundary problem, and the connection with the lattice pattern for a doubly-periodic function is obvious. Rosenhead and others have made important applications of this idea to the study of the behaviour of an aerofoil in a wind-tunnel. The last chapter contains some miscellaneous results ; of these, the connection of elliptic functions with the Tschebyscheff polynomials

$$T_n(x) = \cos\,(n \operatorname{arc} \cos x)$$

and their minimum approximation properties, was, to the reviewer, new and most interesting. T. A. A. B.

Les équations différentielles de la technique. Par C. Blanc. Pp. 314. Fr. sw. 29.50. 1947. (Griffon, Neuchâtel)

This clearly written and clearly printed volume contains the substance of Professor Blanc's lectures to second and third year engineers at the École Polytechnique of the University of Lausanne. The author deals mainly with linear differential equations, ordinary and partial, and nearly always with those whose coefficients are constants. His aim being utilitarian, he is concerned not so much with differential equations and their applications as with problems in technology which lead to soluble differential equations; and while he recognises the need for rigorous proofs, he claims the right to deal with an occasional delicate point by a statement of what can, if necessary, be proved.

The first section, on ordinary equations, is less concerned with writing down complete primitives than with calculating solutions satisfying certain initial or boundary conditions. Thus instead of thinking in terms of complementary functions and particular integrals, the reader is led from free vibrations to transients, steady states, and finally to boundary problems, where the subject begins to be really interesting with its need for Fourier series, Green's functions, eigenvalues and integral equations. Naturally these advanced topics receive only brief references, but each chapter in the book contains indications of further suitable reading.

The second section, planned on the same lines, deals with partial differential equations, D'Alembert's equation, the telegraphic equation, the wave equation, the equation of heat conduction, Poisson's equation. All this in 80 pages means that the treatment can nowhere go very deep, but it is always lucid. As in the first section, judicious use is made of the operational calculus, the author favouring, as perhaps most of us do nowadays, a presentation by means of Laplace transforms.

The third section, of 70 pages, is an appendix of three chapters. That on the calculus of variations omits proofs, but applies Euler's equation to a number of special problems. The account of elliptic functions and integrals is not very illuminating, the author himself supplying the reason when he says that a treatment restricted to real variables can not be comprehensive or rigorous. The outline of the chief properties of Bessel functions and of problems in which cylindrical boundaries or special recurrence conditions call for Bessel functions is much more satisfactory.

In spite of the wealth of technical application, Professor Blanc's course might be thought too academic for engineering students in this country, unless the opinion expressed at a recent meeting of the Mathematical Association, that mathematics must play a much more important part in the curriculum for undergraduate and post-graduate engineers, is more generally accepted. The clarity of exposition and the choice of material in this book should make it valuable to the lecturer. For the student who will ultimately require some really serious mathematics for his technical problems, Professor Blanc's book would serve as a valuable lead in tackling, say, the wealth of methods and results in Harry Bateman's classic treatise. T. A. A. B.

Les fondements psycho-linguistiques des mathématiques. By G. Mannoury. Pp. 63. Fr. sw. 6.60. 1947. (Editions du Griffon, Neuchâtel)

The publication of this French translation of Mannoury's essay on the foundations of mathematics marked the occasion of the author's eightieth birthday.

In the fifteen years which have passed since this account of the principles was written, Mannoury has applied his psycho-linguistic technique in such

diverse fields as aesthetics, ethics and pedagogy, but the attitude to mathematics taken in this introductory essay has not changed.

Psycho-linguistics is defined as the study, by empirical methods, of the network of mental associations which is at the root of an act of communication. The truths of logic and mathematics are held to be a synthesis of our perception of regularities in the workings of our mind and in our experiences of the outside world.

From the standpoint of psychology, Mannoury says (on p. 45) established habits of speech (like the laws of logic) have the same character as natural laws. " There are two essentially different ways (p. 52) of verifying that 13 and 7 make 20 ; by asking questions or by working an abacus ; in the first we verify a linguistic regularity and in the second a natural law." While it is clear that the act of verifying on an abacus that 13 and 7 make 20 *might* be regarded as verifying a law of nature, the law that every time we thread seven beads on to an abacus containing thirteen we completely fill the first wire, yet is this the usual use of an abacus? Counting on an abacus is not, it seems, a physical experiment but a *transformation of number signs*. An abacus is a representation of the idea of a positional notation ; few of us have an immediate perception of thirteen or twenty (in the way we have of two or three)—and no one perceives the number of a large enough group—and so we transform number signs into a notation which demands less of our powers of perception. To suppose that the meaning of " 13 and 7 make 20 " is that everyone says it is so when we question them is, surely, to sacrifice reason on the altar of memory. The parrot and the mathematician may say the same thing, but the mathematician can also prove it.

Mannoury's essay is full of stimulating ideas ; even a reader completely without sympathy for, or interest in, psycho-linguistics, provided he can endure to be told that his belief in the truth and eternity of mathematics is pure superstition, will find in it much to entertain him. R. L. GOODSTEIN.

The Symbolic Method of Vector Analysis, the j Operator simply explained. By W. H. MILLER. Pp. 28. 3s.

A. C. Network Analysis by Symbolic Algebra. By W. H. MILLER. Pp. 41. 4s.

Hyperbolic Functions, their Derivation and Applications in Vector Algebra. By C. A. GROVER. Pp. 40. 4s. Classifax Series, 101, 102, 103. (Classifax Publications, Manchester)

These three short monographs are attractively produced, but they are dear, seeing that each topic is dealt with quite as well in a single chapter of many textbooks on mathematics for engineering students. Presumably they are intended for the engineering student meeting the subjects for the first time. If so, he must be bewildered by some of the statements made early in them, such as the following : " It should be understood that all operations with vector quantities must be carried out vectorially. Thus $V_p = v_1 \times v_2$ indicates that a product must be obtained vectorially " ; this comes from p. 2 of the first monograph before vectors or complex numbers have been explained at all. " Strictly speaking therefore resistance, reactance and impedance are really scalar quantities, but it is convenient and permissible to treat them as vector operators, since when unit current is passed through an impedance the voltage which appears across it is a vector quantity " ; this comes from p. 4 of the second monograph.

The last monograph starts by defining hyperbolic functions with reference to the hyperbola, and so it is not till p. 13 that the exponential forms appear.

Although the connection with the hyperbola is very interesting, one feels that it would have been better in a book for engineers to have started with the exponential definitions, and have used the pages thus saved for some engineering applications of hyperbolic functions, of which there are none at present.

H. V. LOWRY.

Mechanics. I-IV. By P. GANT. Pp. viii, 440, x, vi, x. Parts I and II, 5s. Parts I–III, 7s. 6d. Part III, 3s. 6d. Part IV, 5s. 6d. 1949. (Bell)

This book " is intended for the beginner in mechanics whether he is in his School Certificate year or in his Sixth Form ". Parts I, II, III are designed to cover the Additional Mathematics of the School Certificate examination, but " the book as a whole is intended as a Higher Certificate textbook ".

The first three parts deal with kinematics, dynamics and statics respectively. In kinematics, besides the usual discussion of speed, acceleration, motion in a straight line with constant acceleration and relative velocity, all of which are dealt with in attractive fashion, there is included a chapter on projectiles. In this chapter, as throughout the book, there is a praiseworthy attempt to keep formulae in the background; the author " is firmly convinced that one cannot learn mechanics by learning formulae, and that the idea is more important than the rule to which it leads ".

Newton's laws of motion are introduced in Part II; gravitational units are first used, but in a later chapter absolute units are discussed and employed to solve the examples previously solved using $P/W = a/g$. In dealing with examples the author always gives the " solution " followed by a " discussion "; the latter is usually most valuable and an excellent feature of the book, but occasionally most of the discussion could better come before the solution. In one discussion it is well stated: " It is an important general rule that in a diagram showing forces none shall be omitted "—it is only rarely that the author himself fails to live up to this. Other topics dealt with in Part II are work, energy, power and impulses.

The statics of Part III is perhaps the least satisfactory part of the book. It is based, adequately enough, upon " three separate experimental facts, the Triangle of Forces, the Principle of Moments, and what the writer has termed the Principle of Parallel Forces ". The explanation of this three-fold basis is well-designed, but to one reader at least it seems to produce unnecessary complexity. The resolved part of a force is also established experimentally and is defined as " The total effect of a force in a direction inclined at an angle to its line of action is called its resolved part in that direction ". What is meant by " total effect " and what does it suggest to the reader? It can, one suspects, only lead to confusion. Again, in dealing with simple machines the impression is given that energy is only lost in doing work against frictional forces, whereas work done in moving parts of the machine might be considerable. Thus a smooth-running machine is regarded as 100 per cent. efficient. Consequently, the solution of the example on p. 244 is wrong; it would seem more direct in this problem to find the work done on the load and the work done by the effort, and then equate the ratio of these two quantities to the efficiency in accordance with the definition.

The more advanced topics comprising Part IV are frameworks, general conditions of equilibrium, couples, shearing force and bending moments, circular motion, simple harmonic motion and the elements of the dynamics of a rigid body. All of these are well illustrated by examples. There is then a chapter headed " Experimental Laws " which, somewhat belatedly, deals with Hooke's Law and Newton's Experimental Law of Restitution. The final chapter is an excellent summary of the basic principles and concepts

used in the book, the sort of stock-taking and consideration of fundamental issues which should be undertaken at the end of every such course on mechanics, for it is then, rather than earlier, that such a basic enquiry can be made.

Young teachers will find this book in many ways an excellent guide ; those older will appreciate the experience on which much of the clear exposition is based. For pupils preparing for the Additional Mathematics of the School Certificate examinations the first three parts should be more than adequate, whilst those proceeding to the Higher School Certificate should find the whole a useful text, except that hydrostatics is not included. J. Topping.

A Classbook of Algebra. By S. F. Trustram. Pp. 338. 6s. 9d. With Answers, 8s. Pt. 1 (no Answers), 4s. Pt. 2 (no Answers), 4s. 6d. Teachers' Notes and Answers, pp. 48 + 93 pages of Answers, 4s. 6d. 1949. (Bell)

Part I of this book consists of sets of exercises only ; there are no explanations or worked examples. In Part 2 there are short explanations and worked examples are included. One advantage of this method is that it is possible to produce the book at a comparatively reasonable price. This will appeal to many teachers at a time when book prices have risen much more than allowances for buying them. On the other hand, the monotonous way in which sets of examples follow one another without a break is rather forbidding. The subject is dealt with in an exhaustive way (there are 273 sets of examples besides revision papers, and the answers occupy 93 pages). Each set of examples has a heading clearly describing its purpose, so that it is easy for a teacher to select what he wants. Even in problems on simple equations, there is a set of examples headed " Ages ", another headed " Perimeters ", another " Speed and Travel ", another " Numbers ", and so on. The subject is developed by easy stages, and one difficulty is tackled at a time so that teachers dealing with backward pupils will find this a help.

The book closes with about a dozen pages on Calculus, so as to cover that section of the Alternative Syllabus. In one sense this seems to be put there as an afterthought. The general impression given is that stress is placed more on manipulative work than on Functionality, so that it would seem that the author was thinking mainly of the Ordinary Syllabus. Even for this some of the manipulative work is overdone. It seems unnecessary to have examples with long square roots and equations with four binomial denominators prime to each other.

The Notes, for the most part, consist of short remarks on the Exercises and the subject-matter, although some of the topics are dealt with in greater detail. The author gives the following explanation of subtraction of directed numbers: $(+4)-(+3)$ means " taking away the distance that the point marked $+3$ is from 0 from the distance that the point marked $+4$ is from 0 ". I cannot see how anyone can make any sense of this.

On the subject of the solution of quadratic equations by formula, the author is in favour of the use of the formula without undue delay. I agree that the formula should be used in years subsequent to that in which the solution by completing the square is introduced, but for reasons different to those advanced by the author. The subject is important enough to demand reasonable reliability in the answers given by solvers. Experience has shown that the results based on the method of completing the square are notoriously unsatisfactory. An experimental investigation of this point would be valuable. I suggest that three weeks' drill with the method of completing the square should be followed by a test of, say, four examples per pupil. I think that the number of successes would work out something like this: A forms 40%, B forms 25%, C forms 10%, D forms 5% or under. S. I.

Post-Primary Mathematics. By S. H. Cracknell and F. J. Shonell. Book I. Pp. 132. 4s. 6d. 1949. (Oliver & Boyd)

Teachers of mathematics have, of late, been asked to take books on trust. This is Book I of x books on post-primary mathematics. The content and treatment of the remaining $(x-1)$ books are unknown to the reader, who must therefore guess as to their probable content and value.

This book appears to be intended for the use of Modern School pupils. The authors, in their preface, urge the importance of breaking new ground without undue delay and, while recognising the importance of revision, are against damping enthusiasm by spending too much time on this. They advocate breaking new ground at the outset, and this is psychologically sound.

The authors' aim is to present the subject of mathematics as a unified whole rather than as a number of isolated subjects. They start with the cube and round it develop such ideas as solids, square and right angle in geometry, ab, a^2, abc, etc., in algebra, and volume in arithmetic. The more general cuboid would probably have been better than the cube for this purpose. This weaving together of the different branches of the subject cannot, however, be kept up, and, on the whole, the tendency is to interleave chapters of the various branches of the subject without fusing them into a homogeneous whole. This does not mean that the authors have been less successful in their aim than others. In fact, they have been more successful than most.

Revision of previous work often comes in incidentally, and this is good planning. For instance, there are many examples on multiplication of fractions in dealing with the area of rectangles. Nevertheless, there are prominent gaps in essential revision, such as addition of fractions and decimals, multiplication of decimals, and many other topics.

The worst of a chapter of, say, geometry here and another there is that it is easy for the subject to appear disconnected. On page 21 we are told that an angle is " the space between two lines ", on page 70 that it is the " amount of rotation ", but the protractor is introduced on page 49 followed by exercises in measuring angles. Set-squares are mentioned on page 2 before right angles are discussed. There is no clear justification for this.

The desire to use simple language, which is often applied effectively, is sometimes overdone. On page 28 the authors speak of vertical lines when they mean perpendicular lines, and on page 26 they describe a cuboid as a " solid which has 6 faces not all squares ".

Trigonometry is introduced early in the book and the use of the tangent is developed. There are many suggestions for outdoor work, and instructions are given for making simple surveying instruments. Pupils will find much in this book on which to bite. S. I.

Mathematics for Modern Schools. Book 3. By T. H. Ward Hill. Pp. 195. 6s. 1949. (Harrap)

The pictures which were so attractive a feature of Books 1 and 2 are as good as ever. The Arithmetic deals with everyday matters, such as road accidents, travel by cycle and by train, sales, Post Office Savings Bank, and National Savings Certificates. The use of a table of squares and a ready reckoner are explained. Ratio is discussed, and also specific gravity.

The one chapter on Algebra starts with revision. In the worked equations on page 77 the $\therefore$ sign has been omitted in front of each line ; this is surely bad style. Directed numbers are introduced, and the chapter closes with some applications. The explanations for the rules of multiplication of directed numbers cannot be considered satisfactory. In applying the illustration " distance = number of steps × length of step ", the author does not succeed

in making the "number of steps" a directed number. He could have obtained a negative number of steps if he had taken steps before zero time.

The new parts in Geometry deal mainly with similar triangles. The subject is well developed, and there are good illustrations and applications to surveying and the solution of problems on the lines of scout methods. The Theorem of Pythagoras is also dealt with. S. I.

A Junior Arithmetic. By J. E. Filby. Pp. iv, 155. 5s.; with Answers, 5s. 6d. 1949. (Murray)

In this little book one of the author's aims is to avoid an over-abundance of examples, and he succeeds in this. Two-thirds of the book is taken up with elementary processes, such as the Four Rules, Practice, Fractions and Decimals. Most of the remainder deals with simple rectilineal areas and volumes. The book closes with a chapter on easy percentages, followed by one on Ratio and Proportion. Why this should come after and not before percentages I cannot tell. There are two sets of Problem Papers, each consisting of 12 papers of 5 questions, and these are in the nature of revision. The book is suitable for first year pupils in Grammar Schools. S. I.

IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY.

Summer School in Relaxation Methods.

29th August–22nd September, 1950.

In the long vacation of the years 1945–8 Summer Schools in Relaxation Methods were held at Imperial College, and in 1949 at Michigan University, U.S.A. Their success has encouraged the provision of similar courses in both England and U.S.A. in the coming long vacation, and at Imperial College this is now planned for the four weeks 29th August–22nd September, 1950.

This course will cover the numerical solution of linear algebraic equations, framework problems, Laplace's and Poisson's equations, the biharmonic equation, eigen-value problems, the heat-conduction equation, etc.

The course will consist of daily lectures at 10.15 a.m. with numerous examples to be solved under supervision. To meet the convenience of those who enrol, it is proposed that lectures be given on Tuesdays–Fridays only, (but if requested) facilities for practical work will also be provided on Mondays.

The fee for the course will be £5, payable to the Imperial College. The College will endeavour to provide accommodation in its Hostel building; but it may prove necessary to restrict the number so accommodated. The daily charge for a room is 9s. Meals are obtainable in the College Refectory.

Replies to this circular should be addressed to D. N. de G. Allen, Imperial College, London, S.W. 7. Separate application should be made by each individual, stating (1) the amount of time that can be given to the course, (2) whether accommodation is desired, and if so (3) whether for the whole period or for that period excluding week-ends.

Enquiries concerning the course in U.S.A., which will occupy six weeks in June and July, should be addressed directly to Professor D. H. Pletta at Virginia Polytechnic Institute, Blacksburg, Va., where the course will be held.

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